ROBBINS'S NEW PLANE AND SOLID GEOMETRY

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ROBBINS'S NEW PLANE AND SOLID GEOMETRY

BY

EDWARD RUTLEDGE ROBBINS, A.B.

FORMERLY OF LAWRENCEVILLE SCHOOL

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ROBBINS'S NEW PLANE AND SOLID GEOMETRY.
E. P. 2

FOR THOSE WHOSE PRIVILEGE

IT MAY BE TO ACQUIRE A KNOWLEDGE OF

GEOMETRY

THIS VOLUME HAS BEEN WRITTEN

AND TO THE BOYS AND GIRLS WHO LEARN THE ANCIENT SCIENCE

FROM THESE PAGES, AND WHO ESTEEM THE POWER

OF CORRECT REASONING THE MORE

BECAUSE OF THE LOGIC OF

PURE GEOMETRY

THIS VOLUME IS DEDICATED

PREFACE

THIS "New Plane and Solid Geometry" is not only the outgrowth of the author's long experience in teaching geometry, but has profited further by suggestions from teachers who have used Robbins's "Plane and Solid Geometry" and by the recommendations of the "National Committee of Fifteen." While many new and valuable features have been added in the reconstruction, yet all the characteristics that met with widespread favor in the old book have been retained.

Among the features of the book that make it sound and teachable may be mentioned the following:

1. The book has been written for the pupil. The objects sought in the study of Geometry are (1) to train the mind to accept only those statements as truth for which convincing reasons can be provided, and (2) to cultivate a foresight that will appreciate both the purpose in making a statement and the process of reasoning by which the ultimate truth is established. Thus, the study of this formal science should develop in the pupil the ability to pursue argument coherently, and to establish geometric truths in logical order. To meet the requirements of the various degrees of intellectual capacity and maturity in every class, the reason for every statement is not printed in full but is indicated by a reference. The pupil who knows the reason need not consult the paragraph cited; while the pupil who does not know it may learn it by the reference. It is obvious that the greater progress an individual makes in assimilating the subject and in entering into its spirit, the less need there will be for the printed reference.

2. Every effort has been made to stimulate the mental activity of the pupil. To compel a young student, however, to supply his own demonstrations frequently proves unprofitable as well as

arduous, and engenders in the learner a distaste for a study in which he might otherwise take delight. This text does not aim to produce accomplished geometricians at the completion of the first book, but to aid the learner in his progress throughout the volume, wherever experience has shown that he is likely to require assistance. It is designed, under good instruction, to develop a clear conception of the geometric idea, and to produce at the end of the course a rational individual and a friend of this particular science.

3. The theorems and their demonstrations — the real subject matter of Geometry — are introduced as early in the study as possible.

4. The simple fundamental truths are explained instead of being formally demonstrated.

5. The original exercises are distinguished by their abundance, their practical bearing upon the affairs of life, their careful gradation and classification, and their independence. Every exercise can be solved or demonstrated without the use of any other exercise. Only the truths in the numbered paragraphs are necessary in working originals.

6. The exercises are introduced as near as practicable to the theorems to which they apply.

7. Emphasis is given to the discussion of original constructions.

8. The historical notes give the pupil a knowledge of the development of the science of geometry and add interest to the study.

9. The attractive open page will appeal alike to pupils and to teachers.

10. The Solid Geometry formulas are grouped together at the end of the text, as a ready means of reference.

The author desires to extend his sincere thanks to those friends and fellow teachers who, by suggestion and encouragement, have inspired him in the preparation of the text, as well as to Clement B. Davis for his original and skillful treatment of the illustrations.

EDWARD R. ROBBINS.

CONTENTS

PAGE

Introduction

Definitions 1
Angles 2
Triangles 4
Congruence 5
Symbols 6
Axioms 6
Postulates 7
Exercises 9

Book I. Angles, Lines, Rectilinear Figures

Preliminary Theorems . . . 13
Theorems and Demonstrations 15
Triangles 15
Parallel Lines 20
Quadrilaterals 47
Polygons 60
Symmetry 65
Concerning Original Exercises 68
Summary. General Directions for attacking Originals . 68
Original Exercises 70

Book II. The Circle

Definitions 75
Preliminary Theorems . . . 77
Theorems and Demonstrations 78
Summary 94
Original Exercises 95

PAGE

Kinds of Quantities. Measurement 96
Original Exercises 109
Loci 114
Original Exercises on Loci . 115
Construction Problems . . 117
Analysis 131
Original Construction Problems 132

Book III. Proportion. Similar Figures

Definitions 143
Theorems and Demonstrations 144
Concerning Originals . . . 175
Original Exercises (Numerical) 176
Summary 180
Original Exercises (Theorems) 180
Construction Problems . . 186
Original Construction Problems 190

Book IV. Areas

Definitions 193
Theorems and Demonstrations 193
Formulas 207
Original Exercises (Numerical) 210
Construction Problems . . 213
Original Construction Problems 221

PAGE

Book V. Regular Polygons. Circles

Theorems and Demonstrations 225
Original Exercises (Theorems) 239
Construction Problems . . 242
Formulas 245
Original Exercises (Numerical) 248
Original Construction Problems 253
Maxima and Minima . . . 254
Original Exercises 259

Book VI. Lines, Planes, and Angles in Space

Definitions 261
Preliminary Theorems . . . 262
Theorems and Demonstrations 264
Original Exercises 287
Dihedral Angles 290
Original Exercises 303
Polyhedral Angles 305
Original Exercises 312

Book VII. Polyhedrons

Definitions 313
Prisms 314
Preliminary Theorems. . . 316
Theorems and Demonstrations 317
Original Exercises 330
Pyramids 332
Preliminary Theorems. . . 333

PAGE

Theorems and Demonstrations 334
Regular and Similar Polyhedrons 353
Original Exercises 360

Book VIII. Cylinders and Cones

Cylinders 367
Preliminary Theorems. . . 368
Theorems and Demonstrations 369
Original Exercises 376
Cones 377
Preliminary Theorems. . . 379
Theorems and Demonstrations 381
Original Exercises 388

Book IX. The Sphere

Definitions 393
Preliminary Theorems. . . 395
Theorems and Demonstrations 397
Construction Problems . . 404
Spherical Triangles 408
Preliminary Theorems . . . 410
Theorems and Demonstrations 411
Original Exercises 424
Spherical Areas and Volumes 427
Preliminary Theorems. . . 429
Theorems and Demonstrations 431
Original Exercises 443
Summary of Formulas . . . 451
Index of Definitions . . . 453

PLANE GEOMETRY

INTRODUCTION

1. **Geometry** is a science which treats of the properties of magnitudes.

2. A **point** is that which has position but not magnitude.

3. **A line** is that which has length but no other magnitude.

4. A **straight line** is a line which is determined (fixed in position) by *any* two of its points. That is, two lines that coincide entirely, if they coincide at *any* two points, are straight lines.

5. A **rectilinear figure** is a figure containing straight lines and no others.

6. A **surface** is that which has length and breadth but no other magnitude.

7. A **plane** is a surface in which if any two points are taken, the straight line connecting them lies wholly in that surface.

8. **Plane Geometry** is a science which treats of the properties of magnitudes in a plane.

9. A **solid** is that which has length, breadth, and thickness. A solid is that which occupies space.

10. **Boundaries.** The boundaries (or boundary) of a solid are surfaces. The boundaries (or boundary) of a surface

are lines. The boundaries of a line are points. These boundaries can be no part of the things they limit. A surface is no part of a solid; a line is no part of a surface; a point is no part of a line.

11. Motion. If a point moves, its path is a line. Hence, if a point moves, it generates (describes or traces) a line; if a line moves (except upon itself), it generates a surface, if a surface moves (except upon itself), it generates a solid.

NOTE. Unless otherwise specified the word "line" means *straight line.*

ANGLES

ANGLE | ADJACENT ANGLES | VERTICAL ANGLES | RIGHT ANGLES PERPENDICULAR

12. A **plane angle** is the amount of divergence of two straight lines that meet. The lines are called the **sides** of the angle. The **vertex** of an angle is the point at which the lines meet.

13. Adjacent angles are two angles that have the same vertex and a common side *between them.*

14. Vertical angles are two angles that have the same vertex, the sides of one being prolongations of the sides of the other.

15. If one straight line meets another and makes the adjacent angles equal, the angles are **right angles.**

16. One line is **perpendicular** to another if they meet at right angles. Either line is perpendicular to the other. The point at which the lines meet is the **foot** of the perpendicular. **Oblique lines** are lines that meet but are not perpendicular.

17. A **straight angle** is an angle whose sides lie in the same straight line, but extend in opposite directions from the vertex.

OBTUSE ANGLE | ACUTE ANGLE | COMPLEMENTARY ANGLES | SUPPLEMENTARY ANGLES

18. An **obtuse angle** is an angle that is greater than a right angle. An **acute angle** is an angle that is less than a right angle. An **oblique angle** is any angle that is not a right angle.

19. Two angles are **complementary** if their sum is equal to *one* right angle. Two angles are **supplementary** if their sum is equal to *two* right angles. Thus, the complement of an angle is the difference between one right angle and the given angle. The supplement of an angle is the difference between two right angles and the given angle.

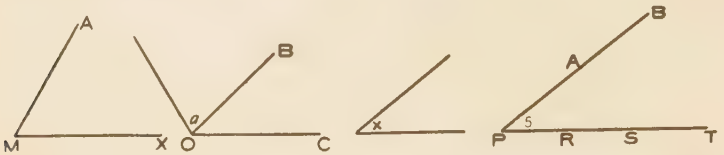
20. A **degree** is one ninetieth of a right angle. The degree is the familiar unit used in measuring angles. It is evident that there are 90° in a right angle; 180° in two right angles, or a straight angle; 360° in four right angles.

There are 60 minutes (60′) in one degree, and 60 seconds (60″) in one minute.

21. **Parallel lines** are straight lines that lie in the same plane and that never meet, however far they are extended in either direction.

22. **Notation.** A point is usually denoted by a capital letter, placed near it. A line is denoted by two capital letters, placed one at each end, or one at each of two of its points. Its length is sometimes represented advantageously by a small letter written near it. Thus, the line AB, the line RS; the line m.

There are various ways of naming angles. Sometimes three capital letters are used, one on each side of the angle and one at the vertex; sometimes a small letter or a figure is placed within the angle. The symbol for angle is $\angle$.



$\angle AMX$ OR $\angle XMA$ OR $\angle M$ $\angle a$ AND $\angle BOC$, NOT $\angle O$ $\angle x$

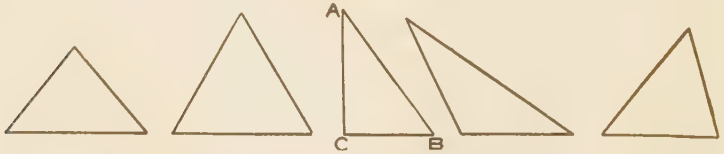
In naming an angle by the use of three letters, the vertex letter is always placed between the others. Thus the $\measuredangle$ above are $\angle AMX$ or $\angle XMA$, $\angle a$, $\angle BOC$, $\angle x$, $\angle APR$, $\angle APS$, $\angle BPR$, $\angle TPB$, $\angle 5$, etc.

In the above figure $\angle x = \angle 5$. The size of an angle depends on the amount of divergence between its sides, and not upon their length.

An angle is said to be **included** by its sides. An angle is **bisected** by a line drawn through the vertex and dividing the angle into two equal angles.

TRIANGLES

23. A **triangle** is a portion of a plane bounded by three straight lines. These lines are the **sides**. The **vertices** of a triangle are the three points at which the sides intersect. The **angles** of a triangle are the three angles at the three vertices. Each side of a triangle has two angles adjoining it. The symbol for triangle is $\triangle$.



ISOSCELES $\triangle$ EQUILATERAL $\triangle$ EQUIANGULAR $\triangle$ RIGHT $\triangle$ OBTUSE $\triangle$ ACUTE $\triangle$ SCALENE $\triangle$

The **base** of a triangle is the side on which the figure appears to stand. The **vertex angle** is the angle opposite the base, and its vertex is **the vertex of the triangle.**

24. Kinds of triangles:

A **scalene** triangle is a triangle **no two sides** of which are equal.
An **isosceles** triangle is a triangle **two sides** of which are equal.
An **equilateral** triangle is a triangle **all sides** of which are equal.
A **right** triangle is a triangle **one angle** of which is a right angle.
An **obtuse** triangle is a triangle **one angle** of which is an obtuse angle.
An **acute** triangle is a triangle **all angles** of which are acute angles.
An **equiangular** triangle is a triangle **all angles** of which are equal.

25. The **hypotenuse** of a right triangle is the side opposite the right angle. The sides forming the right angle are called **legs.**

CONGRUENCE

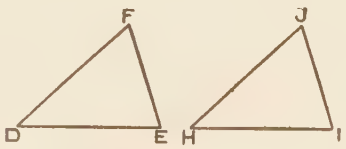
26. Two geometric figures are said to be **equal** if they have the same size or magnitude.

Two geometric figures are said to be **congruent** if, when one is superposed upon the other, they coincide in all respects.

The corresponding parts of congruent figures are equal, and are called **homologous parts.**

27. Homologous parts of congruent figures are equal.

If the triangles DEF and HIJ are congruent,
$\angle D$ is homologous to and = to $\angle H$;
DE is homologous to and = to HI;
$\angle E$ is homologous to and = to $\angle I$;
EF is homologous to and = to IJ.



NOTE. **Congruent figures** have the same shape as well as the same size, whereas **equal figures** do not necessarily have the same shape.

Ex. 1. What is the complement of an angle of 35°? 48°? 80°? 75° 50′? 8° 20′?

Ex. 2. What is the supplement of an angle of 100°? 50°? 148°? 121° 30′? 10° 40′?

28. A **curve** or **curved line**, is a line no part of which is straight.

A **circle** is a plane curve all points of which are equally distant from a point in the plane, called the **center.**

An **arc** is any part of a circle.

A **radius** is a straight line from the center to any point of the circle.

A **diameter** is a straight line containing the center and having its extremities in the circle.

The length of the circle is called the **circumference.**

29. Symbols. The usual symbols and abbreviations employed in geometry are the following:

$+$ plus.	$\odot$ circle.	Ax.	axiom.
$-$ minus.	Ⓢ circles.	Hyp.	hypothesis.
$=$ equals, is equal to, equal.	$\angle$ angle.	comp.	complementary.
$\neq$ does not equal.	$\measuredangle$ angles.	supp.	supplementary.
$\cong$ congruent, *or* is congruent to.	rt. $\angle$ right angle.	Const.	construction.
$>$ is greater than.	rt. $\measuredangle$ right angles.	Cor.	corollary.
$<$ is less than.	$\triangle$ triangle.	st.	straight.
$\therefore$ hence, therefore, consequently.	$\triangleq$ triangles.	rt.	right.
$\perp$ perpendicular.	rt. $\triangleq$ right triangles.	Def.	definition.
$\perp\!\!\!\perp$ perpendiculars.	$\parallel$ parallel.	alt.	alternate.
	$\parallel\!\!\!\parallel$ parallels.	int.	interior.
	$\square$ parallelogram.	ext.	exterior.
	Ⓟ parallelograms.		

AXIOM, POSTULATE, AND THEOREM

30. An **axiom** is a statement admitted without proof to be true.

31. Axioms.

1. Magnitudes that are equal to the same thing, or to equals, are equal to each other.

2. If equals are added to, or subtracted from, equals, the results are equal.

3. If equals are multiplied by, or divided by, equals, the results are equal.

[Doubles of equals are equal; halves of equals are equal.]

4. The whole is equal to the sum of all of its parts.

5. The whole is greater than any of its parts.

6. A magnitude may be replaced by its equal in any process. [Briefly called "substitution."]

7. If equals are added to, or subtracted from, unequals, the results are unequal in the same order.

8. If unequals are added to unequals in the same order, the results are unequal in that order.

9. If unequals are subtracted from equals, the results are unequal in the opposite order.

10. Doubles or halves of unequals are unequal in the same order. Also, unequals multiplied by equals are unequal in the same order.

11. If the first of three magnitudes is greater than the second, and the second is greater than the third, the first is greater than the third.

12. A straight line is the shortest line that can be drawn between two points.

13. Only one line can be drawn through a point parallel to a given line.

14. A geometrical figure may be moved from one position to another without any change in form or magnitude.

32. A **postulate** is the assumption of the possibility of performing a certain geometric operation.

33. POSTULATES.

1. It is possible to draw a straight line from any point to any other point.

2. It is possible to extend (prolong or produce) a straight line indefinitely, or to terminate it at any point.

34. A geometric **proof** or **demonstration** is a logical course of reasoning by which a truth becomes evident.

35. A **theorem** is a statement that requires proof.

In the case of the preliminary theorems which follow, the proof is very simple; but as these theorems are not admitted without proof they cannot be classified with the axioms.

A **corollary** is a truth immediately evident, or readily established from some other truth or truths.

A **proposition**, in geometry, is the statement of a theorem to be proved or a problem to be solved.

Ex. 1. Draw an $\angle ABC$. In $\angle ABC$ draw line BD.
What does $\angle ABD + \angle DBC$ equal?
What does $\angle ABC - \angle ABD$ equal?

Ex. 2. In a rt. $\angle ABC$ draw line BD.
If $\angle ABD = 25°$, how many degrees are there in $\angle DBC$?
How many degrees are there in the complement of an angle of 38°?
How many degrees are there in the supplement?

Ex. 3. Draw a straight line AB and take a point X on it.
What line does $AX + BX$ equal?
What line does $AB - BX$ equal?

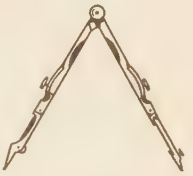
Ex. 4. Draw a straight line AB and prolong it to X so that $BX = AB$.
Prolong it so that $AB = AX$.

Historical Note. Probably as early as 3000 B.C. the Egyptians had some knowledge of geometric truths. The construction of the great pyramids required an acquaintance with the relations of geometry. This knowledge, however vague it may have been, was, according to Herodotus, employed in determining the amount of land washed away by the river Nile, during the reign of Rameses II (1400 B.C.).

The Greeks, however, were the first to study geometry as a logical science. They enunciated theorems and demonstrated them, they propounded problems and solved them as early as 300 B.C., and, in a crude way, two or three centuries earlier. To them belongs the credit of establishing a logical system of geometry that has survived, practically unchanged, for twenty centuries.

EXERCISES EMPLOYING THE TWO INSTRUMENTS OF GEOMETRY

Aside from pencil and paper, the only instruments necessary for the construction of geometrical diagrams are the ruler and the compasses.

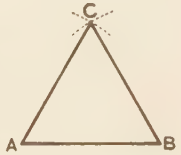


Ex. 1. It is required to draw an equilateral triangle upon a given line as base.

Suppose AB is the given base.

Required to draw an equilateral $\triangle$ upon it.

Using A as a center and AB as a radius, draw an arc. Using B as a center and AB as a radius, draw another arc cutting the first one at C. Draw AC and BC. The $\triangle ABC$ is an equilateral $\triangle$, and AB is its base.

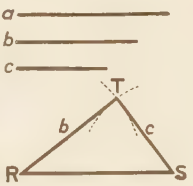


Ex. 2. It is required to draw a triangle having its three sides each equal to a given line.

Suppose the three given lines are a, b, c.

Required to draw a $\triangle$ having for its sides lines equal to a, b, c, respectively.

Draw a line $RS =$ to a. Using R as a center and b as a radius, draw an arc. Using S as a center and c as a radius, draw another arc cutting the first arc at T. Draw straight lines RT and ST. $\triangle RST$ is the $\triangle$ whose three sides are equal to the lines a, b, c, respectively.



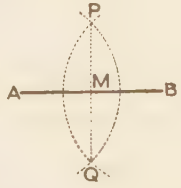
Ex. 3. It is required to find the midpoint of a given straight line.

Given the straight line AB.

Required to find its midpoint.

Using A and B as centers and a radius sufficiently long, draw two arcs, intersecting at P and Q.

Draw the straight line PQ cutting AB at M. Point M is the midpoint of AB.

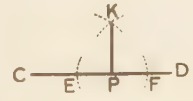


Ex. 4. It is required to draw a perpendicular to a line from a point within the line.

Given the line CD and point P in it.

Required to construct a $\perp$ to CD, at P.

Using P as center and any radius, draw two arcs cutting CD at E and F. Now using E and F as centers and a radius greater than before, draw two arcs intersecting at K. Draw KP. This line KP is $\perp$ to CD at P.



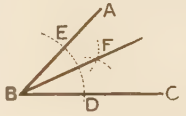
Ex. 5. It is required to draw a perpendicular to a line from a point without the line.

Given line AB and point P, without it.

Required to draw a $\perp$ to AB from P.

Using P as center and a sufficient radius, draw an arc cutting AB at C and D. Now using C and D as centers and a sufficient radius, draw two arcs intersecting at E. Draw PE, meeting AB at R. PR is the required $\perp$ to AB from P.

Ex. 6. It is required to bisect a given angle.



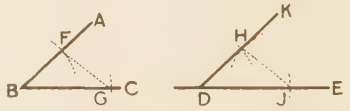
Given the $\angle ABC$.

Required to bisect it.

Using vertex B as a center and any radius, draw arc DE cutting BC at D and BA at E.

Using D and E as centers and a sufficient radius, draw arcs intersecting at F. Draw straight line BF. BF bisects the $\angle ABC$.

Ex. 7. It is required to construct, at a given point on a given line, an angle equal to a given angle.



Given line DE, point D in it, and $\angle B$.

Required to construct an $\angle$ at D, equal to $\angle B$.

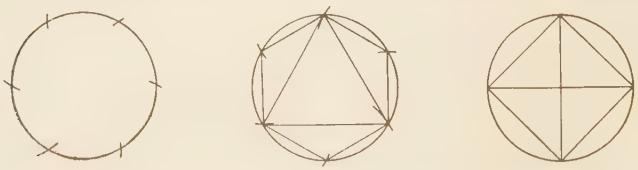
Using B as a center and with any two distances as radii, draw an arc cutting AB at F and another cutting BC at G.

Using D as a center and the same radii as before, draw one arc, and another arc cutting DE at J.

Draw the straight line FG. Using J as a center and FG as a radius, draw an arc cutting a former arc at H. Draw the straight lines HJ and DHK.

Now the $\angle KDE = \angle B$.

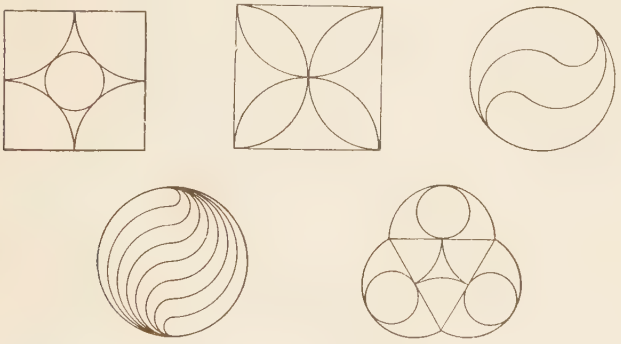
Ex. 8. By the use of ruler and compasses, draw the following figures:



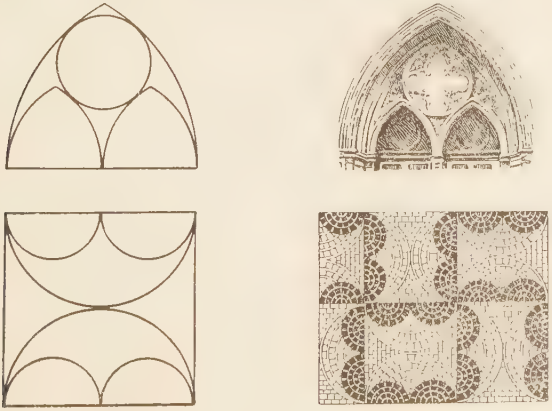
Ex. 9. Does it make any difference in these exercises, which lines are drawn first? In Ex. 7 and Ex. 8 explain the order of the lines drawn.

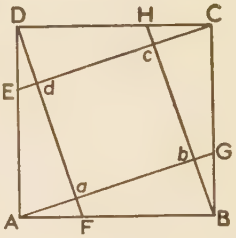
Ex. 10. Using the compasses only, draw the following figures:

Ex. 11. Draw the following figures:

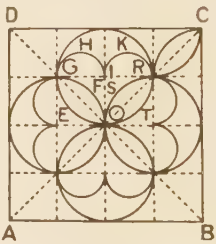


Ex. 12. Draw the first of each of these three pairs of figures. Can you explain the construction of the second figure in each pair?

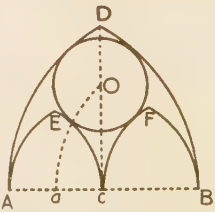




In this figure, *ABCD* is a square. On the sides are measured the equal distances *AE* and *BF*, and *CG* and *DH*; then the lines *AG*, *BH*, *CE*, and *DF*, are drawn intersecting at *a*, *b*, *c*, *d*. The figure *abcd* is also a square. This figure is the basis of an Arabic design used for parquet floors, etc.



In this figure, which is the basis of a mosaic floor design, the radii of all complete circles equal one fourth of the side of the square *ABCD*. The radii of the semicircles *GHI*, *IKR*, etc., equal one eighth of the side of the square.



In this figure *ABD* is an equilateral arch, and *CD* is its altitude. The several centers used are *A*, of arc *BD* and arc *CE*; *B*, of arc *AD* and *CF*; *C*, of arcs *AE* and *BF*.

This figure is the basis of a common Gothic window design.

NOTE. The letters "Q.E.D." are often annexed at the end of a demonstration and stand for "*quod erat demonstrandum*," which means, "which was to be proved."

BOOK I

ANGLES, LINES, RECTILINEAR FIGURES

PRELIMINARY THEOREMS

36. A right angle is equal to half a straight angle.

Because of the definition of a right angle. (15.)

37. A straight angle is equal to two right angles. (36.)

38. Two straight lines can intersect in only one point.

Because they would coincide entirely if they had two common points. (4.)

39. Only one straight line can be drawn between two points. (4.)

40. A definite (limited or finite) **straight line can have only one midpoint.**

Because the halves of a line are equal.

41. All straight angles are equal.

Because they can be made to coincide. (26.)

42. All right angles are equal.

They are halves of straight angles. (36.)

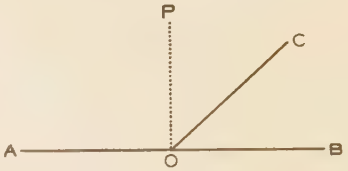
$\therefore$ they are equal. (Ax. 3.)

43. Only one perpendicular to a line can be drawn from a point in the line.

These right angles would not be equal if there were two perpendiculars. (42.)

44. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

Because they together equal two rt. ∡s. (19.)



45. If two adjacent angles are supplementary, their exterior sides are in the same straight line.

Because their sum is two rt. ∡s (19); or a straight ∠ (37). Hence the exterior sides are in the same straight line (17).

46. The sum of all the angles on one side of a straight line at a point equals two right angles. (Ax. 4 and 37.)

47. The sum of all the angles about a point in a plane is equal to four right angles. (46.)

48. Angles that have the same complement are equal. Or, **complements of the same angle, or of equal angles, are equal.**

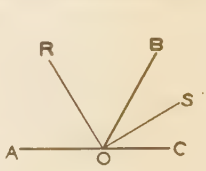
Because equal angles subtracted from equal right angles leave equal angles. (Ax. 2.)

49. Angles that have the same supplement are equal. Or, **supplements of the same angle, or of equal angles, are equal.** (Ax. 2.)

50. If two angles are equal and supplementary, they are right angles.

Each is half a straight ∠; ∴ each is a rt. ∠. (36.)

NOTE. A reference number usually indicates only the statement in **full face type** in the section referred to. In giving demonstrations the pupil should quote the correct reason for each statement.

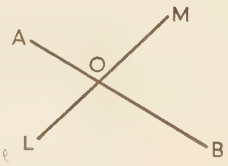


Ex. The bisectors of two supplementary adjacent angles are perpendicular to each other.

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

51. If two straight lines intersect, the vertical angles are equal.



Given: Lines AB and LM intersecting at O, $\measuredangle AOM$ and BOL, a pair of vertical $\measuredangle$.

To Prove: $\angle AOM = \angle BOL$.

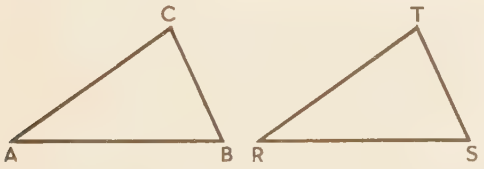
Proof:

$\measuredangle AOM$ and MOB are supplementary	(44).
$\measuredangle MOB$ and BOL are supplementary	(44).
$\therefore \angle AOM = \angle BOL$.	(49.)

Q.E.D.

Proposition II. Theorem

52. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.



Given: $\triangle ABC$ and RST; $AB = RS$; $AC = RT$; $\angle A = \angle R$.

To Prove: $\triangle ABC$ is congruent to $\triangle RST$.

Proof: Place the $\triangle ABC$ upon the $\triangle RST$ so that $\angle A$ coincides with its equal $\angle R$.

AB falls upon RS and point B upon S	$(AB = RS)$.
AC falls upon RT and point C upon T	$(AC = RT)$.
$\therefore BC$ coincides with ST	(39).
$\therefore$ the $\triangle$ coincide and are congruent	(26).

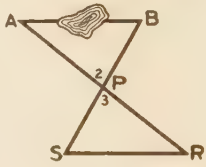
Q.E.D.

53. Corollary. Two right triangles are congruent if two legs of one are equal respectively to two legs of the other.

This is a corollary following immediately from 52.

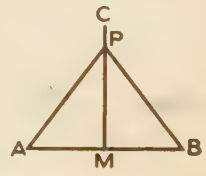
Ex. 1. If two triangles have two sides of one equal to two sides of the other, are the triangles necessarily congruent?

Ex. 2. Illustrate your answer to Ex. 1 by drawing two triangles.



Ex. 3. Find the distance AB if there is an obstruction between A and B.

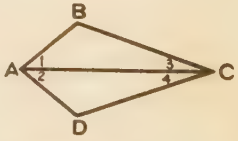
Method. Take a convenient point P, from which A and B are accessible. Measure AP and in same straight line mark point R such that $PR = AP$. Similarly find point S. Show that the two $\triangle ABP$ and SRP are congruent, therefore the length of AB may be found by measuring RS.



Ex. 4. Prove that a point, P, in the perpendicular bisector MC of a line AB is equally distant from the ends of the line AB.

(Show that the $\triangle AMP$ and BMP are congruent, (1) by using 52; and (2) by using 53.)

Ex. 5. If the line AC bisects $\angle BAD$, and $BA = AD$, prove that the triangles ABC and ADC are congruent, the line AC bisects $\angle BCD$, and BC equals CD.



Ex. 6. Prove that if a line from a vertex of a triangle perpendicular to the opposite side bisects that side, the triangle is isosceles.

Ex. 7. Draw two angles that are adjacent and not supplementary; adjacent and not complementary.

Ex. 8. Of two unequal angles which has the greater supplement?

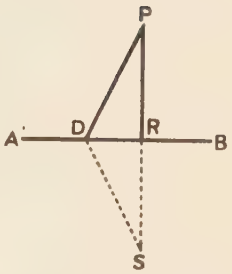
Ex. 9. The complement of a certain angle added to the supplement of the same angle is 176°. Find the angle.

Ex. 10. What angle added to one fifth of its supplement equals a right angle?

Ex. 11. In the figure of 51, if $\angle AOM$ is 100°, how many degrees are there in each of the other angles at O?

PROPOSITION III. THEOREM

54. Only one perpendicular can be drawn to a line from an external point.



Given: $PR \perp$ to AB from P, and PD any other line from P to AB.

To Prove: PD is not $\perp$ to AB; that is, PR is the only $\perp$ from P to AB.

Proof: 1. Extend PR to S making $RS =$ to PR.

2. Draw DS.	
3. In rt. $\triangle$ PDR and SDR, $PR = RS$.	3. Construction.
4. $DR = DR$.	4. Identical.
5. $\therefore \triangle PDR$ is congruent to $\triangle SDR$.	5. Quote 53.
6. $\therefore \angle PDR = \angle SDR$.	6. Quote 27.
7. That is, $\angle PDR =$ half of $\angle PDS$.	
8. Now line PRS is a st. line.	8. Construction.
9. $\therefore$ line PDS is *not* a st. line.	9. Quote 39.
10. $\therefore \angle PDR$, half of $\angle PDS$, is *not* a rt. $\angle$.	10. Quote 36.
11. $\therefore PD$ is *not* $\perp$ to AB.	11. Quote 16.

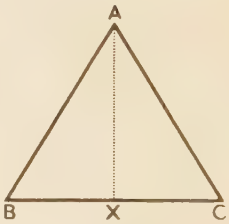
That is, PR is the only $\perp$ from P to AB.

Q.E.D.

The preceding form of demonstration will serve to illustrate an excellent plan of writing the proofs. It will be observed that the *statements* appear at the left of the page and their *reasons* at the right. This arrangement will be found of great value in the saving of time, both for the pupil who writes the proofs and for the teacher who reads them.

Historical Note. The proof of the following theorem as given in the fifth proposition of Euclid's "The Elements," the most famous geometry that was ever written, was considered by the beginners as presenting great difficulties. The theorem was therefore called by the ancient teachers, the *pons asinorum*, or the bridge of the asses. Euclid discussed only magnitudes, not their numerical measures. Another note (p. 45) will tell more of the author of this renowned book.

Proposition IV. Theorem



55. The angles opposite the equal sides of an isosceles triangle are equal.

Given: $\triangle ABC$, $AB = AC$.

To Prove: $\angle B = \angle C$.

Proof: Suppose AX is drawn dividing $\angle BAC$ into two equal $\measuredangle$ and meeting BC at X. In the $\triangle ABX$ and ACX,

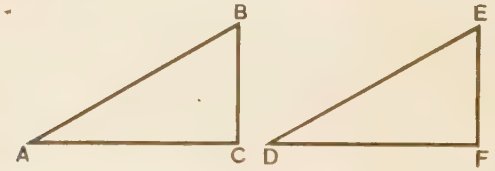
$AX = AX$	(Identical).
$AB = AC$	(Given).
$\angle BAX = \angle CAX$	(Const.).
$\therefore \triangle ABX$ is congruent to $\triangle ACX$	(52).
$\therefore \angle B = \angle C$	(27).

Q.E.D.

56. Corollary. An equilateral triangle is equiangular.

Proposition V. Theorem

57. Two right triangles are congruent if the hypotenuse and an adjoining angle of one are equal respectively to the hypotenuse and an adjoining angle of the other.



Given: Rt. $\triangle ABC$ and DEF; $AB = DE$; and $\angle A = \angle D$.

To Prove: $\triangle ABC$ is congruent to $\triangle DEF$.

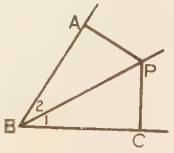
Proof: Place $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and AC falls along DF.

Then AB coincides with DE and point B falls exactly on E ($AB = DE$).

Now, from point E, BC and EF are both $\perp$ to DF (16).
$\therefore$ BC coincides with EF. (54).
$\therefore$ $\triangle ABC$ is congruent to $\triangle DEF$ (26).

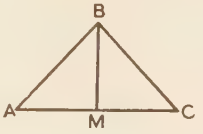
Q.E.D.

Ex. 1. In the adjoining diagram, if $\angle 1 = \angle 2$, prove the right triangles congruent.



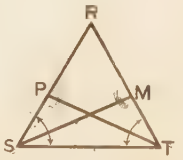
Ex. 2. By use of 27 prove $PA = PC$.

Ex. 3. Prove that the line, BM, from the vertex of an isosceles triangle, ABC, and perpendicular to the base, bisects the vertex angle and also bisects the base.



(First prove the two right $\triangle$ congruent, and then use 27.)

Ex. 4. Prove that the perpendiculars, SM and TP, upon the equal sides of an isosceles triangle, RST, from the opposite vertices, S and T, are equal.



Two ways: (*a*) Show rt. $\triangle PST$ and MST are congruent; or (*b*) show rt. $\triangle RSM$ and RPT are congruent.

Ex. 5. Prove that the bisector of the vertex angle of an isosceles triangle bisects the base at right angles.

58. Homologous parts. Triangles are proved congruent in order that their homologous sides or homologous angles may be proved equal.

59. Auxiliary lines. Often it is impossible to give a simple demonstration without drawing lines not described in the hypothesis. Such lines, used only for the proof, are usually dotted in order to distinguish them from the lines mentioned in the hypothesis and the conclusion.

60. Elements of a theorem. Every theorem contains two parts. The one is assumed to be true; the other results from this assumption. The one part contains the given conditions; the other part states the resulting truth.

The assumed part of a theorem is called the **hypothesis**.

The part of the theorem which is to be proved true is the **conclusion.**

Often the hypothesis is a clause introduced by the word "if." When this conjunction is omitted, the subject of the sentence is known and its qualities, described in the qualifying words, constitute the "given conditions." Thus, in the theorem of 52, the assumed part follows the word "if," and the truth to be proved is: "Two triangles are equal."

The **converse of a theorem** is the theorem obtained by interchanging the hypothesis and the conclusion of the original theorem. Consult 44 and 45, 55 and 114.

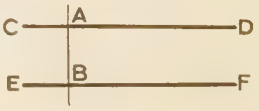
NOTE. Every theorem having a simple hypothesis and a simple conclusion has a converse, but only a few of these converses are true.

61. Elements of a demonstration. All correct demonstrations should consist of certain distinct parts, namely:

1. Full statement of the **given conditions** as applied to a particular figure.
2. Full statement of the truth which it is required **to prove.**
3. The **proof** — a series of successive statements, for each of which a valid reason should be quoted. (The drawing of auxiliary lines is sometimes essential.)
4. The **conclusion** declared to be true.

PROPOSITION VI. THEOREM

62. Two lines in the same plane and perpendicular to the same line are parallel.



Given: *CD* and *EF* in same plane and both ⊥ to *AB*.

To Prove: *CD* and *EF* ∥.

Proof. If *CD* and *EF* were not ∥, they would meet if sufficiently prolonged.

CD and EF are both $\perp$ to AB (Given).

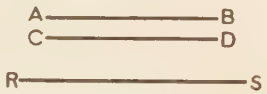
$\therefore$ there would be two lines $\perp$ to AB from the point of meeting. But this is impossible (54).

$\therefore$ CD and EF do not meet and are $\parallel$ (21).

Q.E.D.

Proposition VII. Theorem

63. Two lines in the same plane and parallel to the same line are parallel.



Given: $AB \parallel$ to RS, and $CD \parallel$ to RS, in the same plane.

To Prove: $AB \parallel$ to CD.

Proof: If AB and CD were not $\parallel$, they would meet if sufficiently prolonged.

AB and CD are both $\parallel$ to RS (Given).

$\therefore$ there would be two lines $\parallel$ to RS through the point of meeting. But this is impossible (Ax. 13).

$\therefore$ AB and CD do not meet and are $\parallel$ (21).

Q.E.D.

Proposition VIII. Theorem

64. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.

Given: $LM \perp$ to AB and $AB \parallel$ to CD.

To Prove: $LM \perp$ to CD.

Proof: Suppose XY is drawn through $M \perp$ to LM

XY is $\parallel$ to AB (62).

But CD is $\parallel$ to AB (Given).

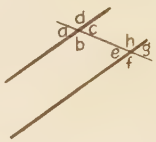
And CD and XY both contain point M (Const.).

$\therefore$ CD and XY coincide (Ax. 13).

But LM is $\perp$ to XY (Const.).

That is, LM is $\perp$ to CD Q.E.D.

65. If one line cuts other lines, it is called a **transversal**. Angles are formed at the several intersections, as follows:



b, c, e, h are **interior angles.**

a, d, f, g are **exterior angles.**

b and h, c and e, a and g, d and f (on opposite sides of the transversal) are **alternate angles.**

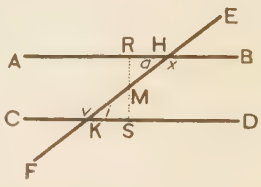
b and h, c and e are **alternate interior angles.**

a and g, d and f are **alternate exterior angles.**

a and e, d and h, b and f, c and g are **corresponding angles.**

Proposition IX. Theorem

66. If a transversal intersects two parallels, the alternate interior angles are equal.



Given: $AB \parallel$ to CD; transversal EF cutting the $\parallel$s at H and K.

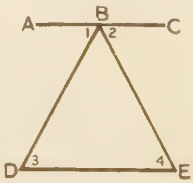
To Prove: $\angle a = \angle i$ and $\angle x = \angle v$.

Proof: Through M, the midpoint of HK, draw $RS \perp$ to AB. Then RS is $\perp$ to CD. (64).

In rt. $\triangle RMH$ and KMS,	$HM = KM$	(Const.).
	$\angle RMH = \angle KMS$	(51).
	$\therefore \triangle RMH$ is congruent to $\triangle KMS$	(57).
	$\therefore \angle a = \angle i$	(27).
Again	$\angle x$ is the supplement of $\angle a$	(44).
Also	$\angle v$ is the supplement of $\angle i$	(44).
	$\therefore \angle x = \angle v$	(49).

Q.E.D.

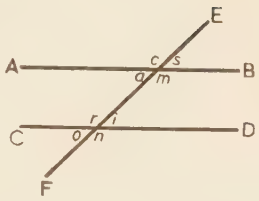
Ex. 1. If a line through the vertex of an isosceles triangle is parallel to the base, it makes equal angles with the sides of the triangle.



Ex. 2. If from each point at which a transversal intersects two parallels a perpendicular to the other parallel is drawn, two congruent right triangles are formed.

Proposition X. Theorem

67. If a transversal intersects two parallels, the corresponding angles are equal.



Given: $AB \parallel$ to CD; transversal EF cutting the $\parallel$s and forming the 8 $\angle$s.

To Prove: $\angle s = \angle i$; $\angle c = \angle r$; $\angle o = \angle a$; $\angle n = \angle m$.

Proof: I.

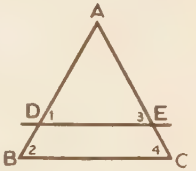
$\angle s = \angle a$	(51).
$\angle a = \angle i$	(66).
$\therefore \angle s = \angle i$	(Ax. 1).

II.

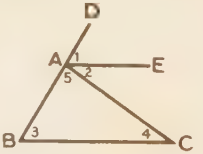
$\angle c = \angle m$	(?).
$\angle m = \angle r$	(?).
$\therefore \angle c = \angle r$	(?).

Etc. Q.E.D.

Ex. 1. If a line intersects the equal sides of an isosceles triangle and is parallel to the third side, the triangle formed has two equal angles.



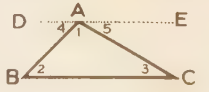
Ex. 2. In the adjoining figure, if AE is parallel to BC, prove that $\angle 1 + \angle 2$ or $\angle CAD = \angle 3 + \angle 4$.



Ex. 3. In the same figure prove that

$$\angle 3 + \angle 4 + \angle 5 = 2 \text{ rt. } \angle\text{s}.$$

Ex. 4. By drawing a line, DE, through the vertex A, of a triangle ABC, parallel to BC, prove that the sum of the angles of any triangle is equal to two right angles.



Ex. 5. If the equal sides of an isosceles triangle are prolonged through the vertex, and a line is drawn parallel to the base, cutting these prolongations, this triangle formed has two equal angles.

Ex. 6. If two sides of any triangle are prolonged beyond the third side, and a line, parallel to the third is drawn cutting these prolongations, two mutually equiangular triangles are formed.

Proposition XI. Theorem

68. If a transversal intersects two parallels, the alternate exterior angles are equal.

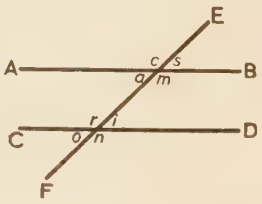
Given: (?) **To Prove:** (?)

Proof: $\angle c = \angle r$ (?); $\angle r = \angle n$ (?). $\therefore \angle c = \angle n$ (?) etc.

Corr. ∠s 25-L. Q.E.D.

Proposition XII. Theorem

69. If a transversal intersects two parallels, the interior angles on the same side of the transversal are supplementary.



Given: (?)

To Prove: $\angle a + \angle r = 2$ rt. ∡s etc.

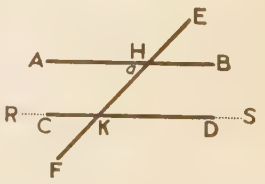
Proof: $\angle a + \angle m = 2$ rt ∡s (46).

But $\angle m = \angle r$ (66).

$\therefore \angle a + \angle r = 2$ rt. ∡s etc. (Ax. 6). Q.E.D.

Proposition XIII. Theorem

70. If a transversal intersects two lines and the alternate interior angles are equal, the lines are parallel. [Converse of 66.]



Given: *AB* and *CD* two lines; transversal *EF* cutting them at *H* and *K*, respectively; $\angle a = \angle HKD$.

To Prove: *CD* ∥ to *AB*.

Proof: Through *K* suppose *RS* is drawn ∥ to *AB*.

Then $\angle a = \angle HKS$ (66).

But $\angle a = \angle HKD$ (Hyp.).

$\therefore \angle HKS = \angle HKD$ (Ax. 1).

That is, *KD* and *KS* coincide, and *CD* and *RS* are the same line. $\therefore$ *CD* is ∥ to *AB*.

(Because it coincides with *RS*, which is ∥ to *AB*.) Q.E.D.

Proposition XIV. Theorem

71. If a transversal intersects two lines and the corresponding angles are equal, the lines are parallel. [Converse of 67.]

Given: AB and CD cut by EF; $\angle c = \angle r$. (Fig. in 69.)

To Prove: $AB \parallel$ to CD.

Proof:

$\angle c = \angle m$	(51).
$\angle c = \angle r$	(Hyp.).
$\therefore \angle m = \angle r$	(Ax. 1).
$\therefore AB$ is $\parallel$ to CD.	(70).

Q.E.D.

Proposition XV. Theorem

72. If a transversal intersects two lines and the alternate exterior angles are equal, the lines are parallel. [Converse of 68.]

Given: AB and CD cut by EF, and $\angle c = \angle n$.

To Prove: $AB \parallel$ to CD.

Proof:

$\angle c = \angle m$	(51).
$\angle c = \angle n$	(Hyp.).
$\therefore \angle m = \angle n$	(Ax. 1).
$\therefore AB$ is $\parallel$ to CD	(71). Q.E.D.

Proposition XVI. Theorem

73. If a transversal intersects two lines and the interior angles on the same side of the transversal are supplementary, the lines are parallel. [Converse of 69.]

Given: AB and CD cut by EF and $\angle a + \angle r = 2$ rt. $\measuredangle$.

To Prove: $AB \parallel$ to CD.

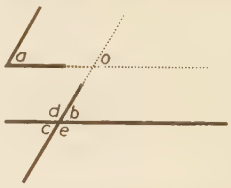
Proof:

$\angle a$ is the supplement of $\angle c$	(44).
$\angle a$ is the supplement of $\angle r$	(Hyp.).
$\therefore \angle c = \angle r$	(49).
$\therefore AB$ is $\parallel$ to CD	(71). Q.E.D.

Proposition XVII. Theorem

74. If two angles have their sides parallel each to each, the angles are equal or supplementary.

Note. There are three cases: (I) the pairs of sides extending in the same two directions from the vertices; (II) the pairs of sides extending in opposite directions from the vertices; (III) one pair extending in the same direction and the other pair in opposite directions from the vertices.



I. **Given:** $\angle a$ and $\angle b$, with their sides $\parallel$ each to each and extending in the *same directions* from their vertices.

To Prove: $\angle a = \angle b$.

Proof: If the non-parallel lines do not meet, produce them to meet, forming $\angle o$.

$\angle a = \angle o$ (67).

$\angle o = \angle b$ (67).

$\therefore \angle a = \angle b$ (Ax. 1).

Q.E.D.

II. **Given:** $\angle a$ and $\angle c$ with their sides $\parallel$ each to each and extending in *opposite directions* from their vertices.

To Prove: $\angle a = \angle c$

Proof:

$\angle a = \angle b$ (Proved in I).

$\angle b = \angle c$ (51).

$\therefore \angle a = \angle c$ (Ax. 1).

Q.E.D.

III. **Given:** $\angle a$ and $\angle d$ with their sides $\parallel$; one pair extending in the *same direction* and the other pair in *opposite directions* from their vertices.

To Prove: $\angle a$ is supplementary to $\angle d$.

Proof: $\angle b$ is supplementary to $\angle d$ (44).

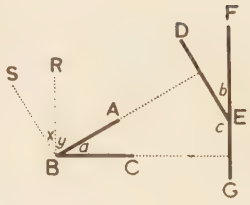
But $\angle a = \angle b$ (Proved in I).

Substituting, $\angle a$ is supp. to $\angle d$. (Ax. 6). Q.E.D.

The proof that $\angle a$ and $\angle e$ are supplementary is the same.

Proposition XVIII. Theorem

75. If two angles have their sides perpendicular each to each, the angles are equal or supplementary.



I. **Given:** $\measuredangle a$ and b with sides $\perp$ each to each.

To Prove: $\angle a = \angle b$.

Proof: At B suppose BR is drawn $\perp$ to BC and $BS \perp$ to AB.

BR is $\parallel$ to FE and BS is $\parallel$ to DE (62).

$\therefore \angle x = \angle b$ (74).

Now $\angle a$ is the complement of $\angle y$ (19).

Also $\angle x$ is the complement of $\angle y$ (19).

$\therefore \angle a = \angle x$ (48).

$\therefore \angle a = \angle b$ (Ax. 1).

II. **Given:** $\measuredangle a$ and c with sides $\perp$ each to each.

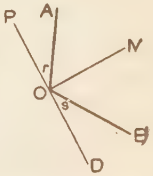
To Prove: $\angle a$ and $\angle c$ supplementary.

Proof: $\angle b$ and $\angle c$ are supplementary (44).

But $\angle a = \angle b$ (Proved in I).

Substituting, $\angle a$ and $\angle c$ are supplementary (Ax. 6).

Q.E.D.



Ex. 1. If a line is drawn through the vertex of an angle and perpendicular to the bisector of the angle, it makes equal angles with the sides.

Ex. 2. Draw figure for Proposition XVII showing $\angle b$ within $\angle a$, and prove the theorem.

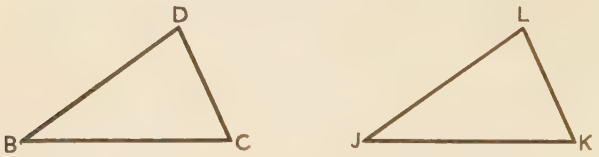
Ex. 3. Draw figure for Proposition XVIII showing $\angle b$ without $\angle a$, and prove the theorem.

Ex. 4. Prove Proposition XVIII if the given angles have the same vertex; if the vertex of one angle is on a side of the other.

Ex. 5. In figure of 75, if $\angle a = 40°$, tell how many degrees there are in $\measuredangle b$, c, x, y, and SBC.

Proposition XIX. Theorem

76. Two triangles are congruent if a side and the two angles adjoining it in the one are equal respectively to a side and the two angles adjoining it in the other.



Given: $\triangle BCD$ and JKL; $BC = JK$; $\angle B = \angle J$; $\angle C = \angle K$.

To Prove: $\triangle BCD$ is congruent to $\triangle JKL$.

Proof: Place $\triangle BCD$ upon $\triangle JKL$ so that BC coincides with its equal, JK.

BD falls on JL (Because $\angle B$ is given = to $\angle J$).

CD falls on KL (Because $\angle C$ is given = to $\angle K$).

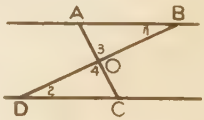
Then point D, which falls on both the lines JL and KL, falls at their intersection, L (38).

$\therefore$ the $\triangle$ are congruent (26).

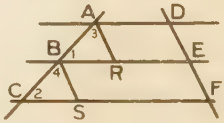
Q.E.D.

77. Corollary. Two right triangles are congruent if a leg and the adjoining acute angle of one are equal respectively to a leg and the adjoining acute angle of the other.

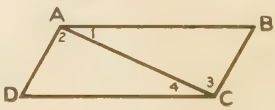
Ex. 1. If AB is $\parallel$ to DC and point O bisects transversal BD, prove that it also bisects transversal AC.



Ex. 2. In the accompanying figure, the three $\parallel$s are cut by two transversals, $AB = BC$, AR and BS are $\parallel$ to DF. Prove that $AR = BS$.

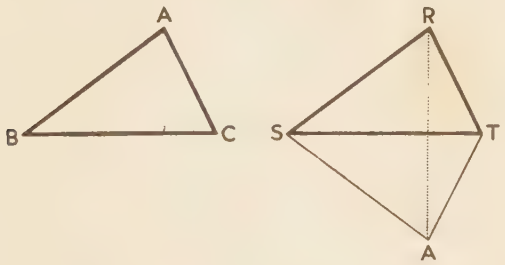


Ex. 3. In the accompanying figure, AB is $\parallel$ to DC and AD is $\parallel$ to BC. Prove that the $\triangle$ are congruent.



Proposition XX. Theorem

78. Two triangles are congruent, if the three sides of one are equal respectively to the three sides of the other.



Given: $\triangle ABC$ and RST; $AB = RS$; $AC = RT$; $BC = ST$.

To Prove: $\triangle RST$ is congruent to $\triangle ABC$.

Proof: Place $\triangle ABC$ in the position of $\triangle AST$, so that the longest equal sides, BC and ST coincide, and A is opposite ST from R. Draw RA.

Now	$RS = AS$	(Hyp.).
	$\therefore \triangle ASR$ is an isosceles $\triangle$	(Def.).
Also	$TR = TA$	(Hyp.).
	$\therefore \triangle ATR$ is an isosceles $\triangle$	(Def.).
	$\therefore \angle SRA = \angle SAR$	(55).
Also	$\angle TRA = \angle TAR$	(55).
Adding,	$\angle SRT = \angle SAT$	(Ax. 2).
	$\therefore \triangle RST$ is congruent to $\triangle AST$	(52).
That is, by substituting, $\triangle RST$ is congruent to $\triangle ABC$		(Ax. 6).

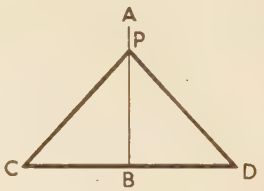
Q.E.D

Ex. In the figure of Ex. 3 on the opposite page, if the opposite sides are equal, prove them parallel.

79. The **distance** from one point to another is the length of the straight line joining the two points.

Proposition XXI. Theorem

80. Any point in the perpendicular bisector of a line is equally distant from the extremities of the line.



Given: $AB \perp$ to CD at its midpoint, B; P any point in AB; PC and PD.

To Prove: $PC = PD$.

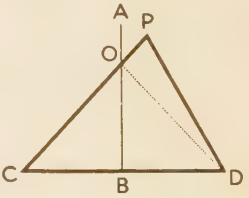
Proof: In the rt. $\triangle PBC$ and PBD,

$PB = PB$	(Identical).
$BC = BD$	(Hyp.).
$\therefore \triangle PBC$ is congruent to $\triangle PBD$	(53).
$\therefore PC = PD.$	(27).

Q.E.D.

Proposition XXII. Theorem

81. Any point not in the perpendicular bisector of a line is not equally distant from the extremities of the line.



Given: $AB \perp$ bisector of CD; P any point not in AB; PC and PD.

To Prove: $PC \neq PD$.

Proof: Either PC or PD will cut AB. Suppose PC cuts AB at O. Draw OD.

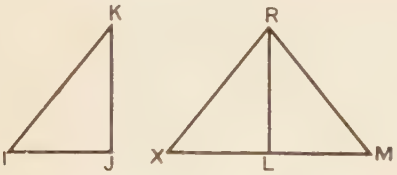
	$DO + OP > PD$	(Ax. 12).
But	$CO = DO$	(80).
Substituting,	$CO + OP > PD$	(Ax. 6).
That is,	$PC > PD$ or $PC \neq PD$.	Q.E.D.

82. Corollary. If a point is equally distant from the extremities of a line, it is in the perpendicular bisector of the line. (80 and 81.)

83. Corollary. Two points each equally distant from the extremities of a line determine the perpendicular bisector of the line. (82 and 4.)

Proposition XXIII. Theorem

84. Two right triangles are congruent if the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.



Given: rt. $\triangle IJK$ and LMR; $KI = RM$; $KJ = RL$.

To Prove: $\triangle IJK = \triangle LMR$.

Proof: Place $\triangle IJK$ in the position of $\triangle XLR$ so that the equal sides, KJ and RL, coincide, and I is at X, opposite RL from M.

Now	$\measuredangle RLM$ and RLX are supplementary	(19).
	$\therefore XLM$ is a straight line	(45).
	$\therefore$ figure XMR is a $\triangle$	(23).
Now	$RX = RM$	(Hyp.).
	$\therefore \triangle XMR$ is isosceles	(Def.).
	$\therefore \angle X = \angle M$	(55).
	$\therefore \triangle XLR$ is congruent to $\triangle LMR$	(57).
That is,	$\triangle IJK$ is congruent to $\triangle LMR$	(Ax. 6).

Q.E.D.

85. Corollary. The perpendicular from the vertex of an isosceles triangle to the base bisects the base, and bisects the vertex angle.

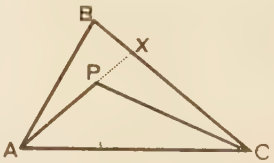
In the congruent right $\triangle$ of 84,	$XL = LM$	(27).
Also	$\angle XRL = MRL$	(27).

Q.E.D.

Ex. In the figure of 84, if $XL = LM$, prove by two methods that $XR = RM$.

Proposition XXIV. Theorem

86. The sum of two sides of a triangle is greater than the sum of two lines drawn to the extremities of the third side, from any point within the triangle.



Given: P, any point in $\triangle ABC$; lines PA and PC.

To Prove: $AB + BC > AP + PC$.

Proof: Extend AP to meet BC at X.

$AB + BX > AP + PX$ (Ax. 12).

$CX + PX > PC$ (Ax. 12).

Adding, $AB + \overline{BX + CX} + PX > AP + PC + PX$ (Ax. 8).

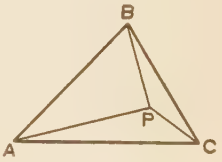
Subtract $PX = PX$

$\therefore AB + BC > AP + PC$ (Ax. 7). Q.E.D.

Ex. If from any point within a triangle lines are drawn to the three vertices:

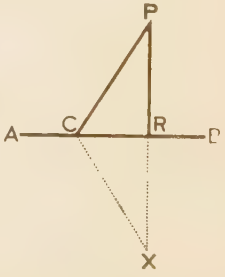
(1) their sum is less than the sum of the sides of the triangle.

(2) their sum is greater than half the sum of the sides of the triangle.



Proposition XXV. Theorem

87. The perpendicular is the shortest line that can be drawn from a point to a straight line.



Given: $PR \perp$ to AB; PC not $\perp$.

To Prove: $PR < PC$.

Proof: Extend PR to X, making $RX =$ to PR. Draw CX.

(1) $PR + RX < PC + CX$ (Ax. 12).

But AR is $\perp$ bisector of PX (Const.).

$\therefore CX = PC$ (80).

Also $RX = PR$ (Const.).

$\therefore$ Substituting in (1), $PR + PR < PC + PC$ (Ax. 6).
That is, $2\,PR < 2\,PC$.
$\therefore PR < PC$ (Ax. 10). Q.E.D.

Proposition XXVI. Theorem

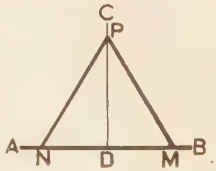
88. If from any point in a perpendicular to a line two oblique lines are drawn,

I. Oblique lines cutting off equal distances from the foot of the perpendicular are equal.

II. Equal oblique lines cut off equal distances. [Converse.]

III. Oblique lines cutting off unequal distances are unequal, and that one which cuts off the greater distance is the greater.

IV. Unequal oblique lines cut off unequal distances from the foot of the perpendicular, and the longer oblique line cuts off the greater distance. [Converse.]



I. **Given:** $CD \perp$ to AB; $ND = MD$; oblique lines PN and PM.

To Prove: $PN = PM$.

Proof: In the rt. $\triangle PDN$ and PDM, $PD = PD$ (Iden.).
Also $ND = DM$ (Hyp.).
$\therefore \triangle PDN$ is congruent to $\triangle PDM$ (53).
$\therefore PN = PM$ (27).
Q.E.D.

II. **Given:** $CD \perp$ to AB; $PN = PM$.

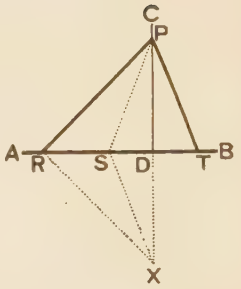
To Prove: $DN = DM$.

Proof: In the rt. $\triangle PDN$ and PDM, $PD = PD$ (Iden.).
Also $PN = PM$ (Hyp.).
$\therefore \triangle PDN$ is congruent to $\triangle PDM$ (84).
$\therefore DN = DM$ (27).
Q.E.D.

III. **Given:** $CD \perp$ to AB; oblique lines PR and PT:

Also $RD > DT$.

To Prove: $PR > PT$.



Proof: Because $DR > DT$, we may take DS (on DR) = to DT. Draw PS. Extend PD to X, making $DX =$ to PD. Draw RX and SX. Now AD is the $\perp$ bisector of PX and CD is the $\perp$ bisector of ST (Const.).

$PR + RX > PS + SX$ (86).

But $RX = PR$, and $SX = PS = PT$ (80).

Substituting, $PR + PR > PS + PS$ (Ax. 6).

That is, $2PR > 2PS$

$\therefore PR > PS$ (Ax. 10).

Substituting, $PR > PT$ (Ax. 6). Q.E.D.

IV. **Given:** [Use no dotted lines.] $CD \perp$ to AB; oblique lines PR and PT; $PR > PT$.

To Prove: $DR > DT$.

Proof: It is evident that $DR < DT$, or $DR = DT$, or $DR > DT$.

1ST: If $DR < DT$, $PR < PT$ (By III).

But $PR > PT$ (Hyp.).

$\therefore DR$ is *not* $< DT$

2D: If $DR = DT$, $PR = PT$ (By I).

But $PR > PT$ (Hyp.).

$\therefore DR$ is *not* $= DT$

3D: Therefore, the only possibility is that $DR > DT$.

Q.E.D.

89. COROLLARY. **From an external point it is not possible to draw three equal straight lines to a given straight line.**

90. The method of demonstration employed in 88, IV, is called the **method of exclusion.**

It consists in making all possible suppositions, leaving the probable one last, and then proving all these suppositions impossible, except the last, which must necessarily be true.

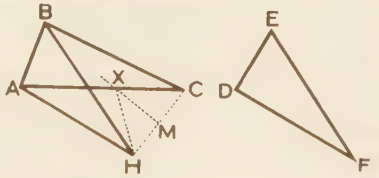
The method of proving the individual steps is called *reductio ad absurdum* (reduction to an absurd or impossible conclusion). This method consists in assuming as false the truth to be proved and then showing that this assumption leads to a conclusion altogether contrary to known truth or the given hypothesis. (Examine the last preceding proof.) This is sometimes called the **indirect method.** The theorems of 62 and 63 are demonstrated by a single use of this method.

Proposition XXVII. Theorem

91. If two triangles have two sides of one equal to two sides of the other, but the included angle in the first greater than the included angle in the second, the third side of the first is greater than the third side of the second.

Given: $\triangle ABC$, DEF; $AB = DE$; $BC = EF$; $\angle ABC > \angle E$.

To Prove: $AC > DF$.



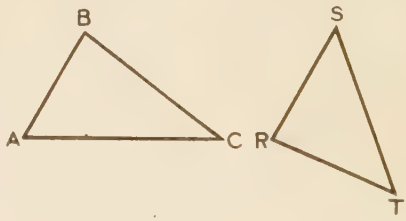
Proof: Place the $\triangle DEF$ upon $\triangle ABC$ so that side DE coincides with its equal AB, $\triangle DEF$ taking the position of $\triangle ABH$. There remains an $\angle HBC$. (Hyp.).

Draw HC and suppose its $\perp$ bisector, MX, to be erected, meeting AC at X. Draw HX.

Now	$HX = XC$	(80).
Also	$AX + XH > AH$	(Ax. 12).
Substituting,	$AX + XC$, or $AC > AH$	(Ax. 6).
Substituting,	$AC > DF$	(Ax. 6). Q.E.D.

Proposition XXVIII. Theorem

92. If two triangles have two sides of one equal to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second. [Converse.]



Given: $\triangle ABC$ and RST; $AB = RS$; $BC = ST$; $AC > RT$.

To Prove: $\angle B > \angle S$.

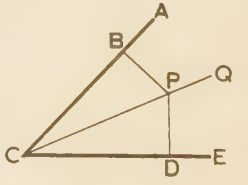
Proof: $\angle B < \angle S$, or $\angle B = \angle S$, or $\angle B > \angle S$.

1. If $\angle B < \angle S$, $AC < RT$ (91).
But $AC > RT$ (Hyp.).
$\therefore \angle B$ is *not* $< \angle S$.
2. If $\angle B = \angle S$, the $\triangle$ are congruent (52).
$\therefore AC = RT$ (27).
But $AC > RT$ (Hyp.).
$\therefore \angle B \neq \angle S$.
3. $\therefore$ the only possibility is that $\angle B > \angle S$. Q.E.D.

93. The **distance** from a point to a line is the length of the perpendicular from the point to the line. If the perpendiculars from a point to two lines are equal, the point is said to be **equally distant** from the lines.

Proposition XXIX. Theorem

94. Every point in the bisector of an angle is equally distant from the sides of the angle.



Given: $\angle ACE$; bisector CQ; point P in CQ; distances PB and PD.

To Prove: $PB = PD$.

Proof: $\triangle PBC$ and PDC are rt. $\triangle$.
In rt. $\triangle PBC$ and PDC, $PC = PC$ (Iden.).
$\angle PCB = \angle PCD$ (Hyp.).
$\therefore \triangle PBC$ is congruent to $\triangle PDC$ (57).
$PB = PD$ (27). Q.E.D.

Proposition XXX. Theorem

95. Every point equally distant from the sides of an angle is in the bisector of the angle.

Given: $\angle ACE$; P, a point, such that $PB = PD$ (distances); CQ, a line from vertex of the angle, and containing P.

To Prove: $\angle ACQ = \angle ECQ$.

Proof: $\triangle PBC$ and PDC are rt. $\triangle$ (93).
In rt. $\triangle PBC$ and PDC, $PC = PC$ (Iden.).
$PB = PD$ (Hyp.)
$\therefore \triangle PBC$ is congruent to $\triangle PDC$ (84).
$\therefore \angle ACQ = \angle ECQ$ (27). Q.E.D.

96. Corollary. If a point is not equally distant from the sides of an angle, it is not in the bisector of the angle.

(If it were in the bisector, it would be equally distant.)

97. Corollary. The vertex of an angle and a point equally distant from its sides determine the bisector of the angle.

98. The **altitude** of a triangle is the perpendicular from any vertex to the opposite side (prolonged if necessary). A triangle has three altitudes.

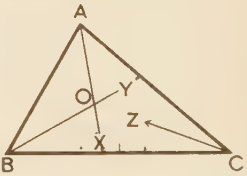
The **bisector** of an angle of a triangle is the line dividing any angle into two equal angles. A triangle has three bisectors of its angles.

The **median** of a triangle is the line drawn from any vertex to the midpoint of the opposite side. A triangle has three medians.

THE THREE MEDIANS

Proposition XXXI. Theorem

99. The bisectors of the angles of a triangle meet in a point which is equally distant from the sides.



Given: $\triangle ABC$, AX bisecting $\angle A$, BY and CZ the other bisectors.

To Prove: AX, BY, CZ meet in a point equally distant from AB, AC, and BC.

Proof: Suppose that AX and BY intersect at O.

O, in AX, is equally distant from AB and AC (94).

O, in BY, is equally distant from AB and BC (?).

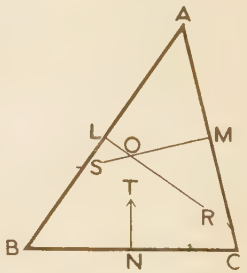
$\therefore$ point O is equally distant from AC and BC (Ax. 1).

$\therefore O$ is in bisector CZ (95).

That is, all three bisectors meet at O, and O is equally distant from the three sides. Q.E.D.

Proposition XXXII. Theorem

100. The three perpendicular bisectors of the sides of a triangle meet in a point which is equally distant from the vertices.



Given: $\triangle ABC$; LR, MS, NT, the three $\perp$ bisectors.

To Prove: LR, MS, NT meet in a point equally distant from A, B, C.

Proof: Suppose that LR and MS intersect at O.

O, in LR, is equally distant from A and B (80).

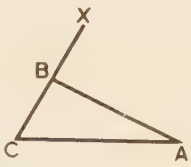
O, in MS, is equally distant from A and C (?).

$\therefore$ point O is equally distant from B and C (Ax. 1).

$\therefore O$ is in $\perp$ bisector NT (82).

That is, all three $\perp$ bisectors meet at O, and O is equally distant from A and B and C. Q.E.D.

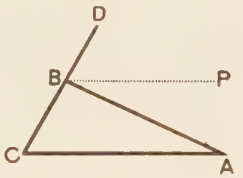
101. An **exterior angle** of a triangle is an angle formed outside the triangle, between one side of the triangle and another side prolonged. [$\angle ABX$.]



The angles within the triangle at the other vertices are the **opposite interior angles.** [$\angle A$ and $\angle C$.]

Proposition XXXIII. Theorem

102. An exterior angle of a triangle is equal to the sum of the opposite interior angles.



Given: $\triangle ABC$; exterior $\angle ABD$.

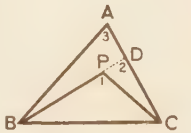
To Prove: $\angle ABD = \angle A + \angle C$.

Proof: Suppose BP to be drawn through $B \parallel$ to AC.

	$\angle ABD = \angle ABP + \angle PBD$	(Ax. 4).
But	$\angle ABP = \angle A$	(66).
Also	$\angle PBD = \angle C$	(67).
	$\therefore \angle ABD = \angle A + \angle C$	(Ax. 6). Q.E.D.

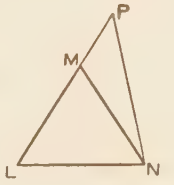
103. Corollary. **An exterior angle of a triangle is greater than either of the opposite interior angles.** (Ax. 5.)

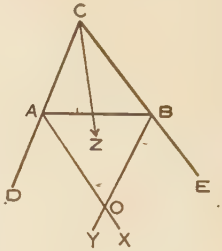
Ex. 1. If lines are drawn from any point within a triangle to two vertices of the triangle, they include an angle greater than the third angle of the triangle.



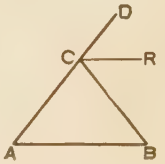
[Notice that $\angle 1$ is an ext. $\angle$ of $\triangle CDP$ and $\angle 2$ is an ext. $\angle$ of $\triangle ABD$.]

Ex. 2. If the side LM, of equilateral triangle LMN, is produced to P, and PN is drawn, $\angle P < \angle L$.





Ex. 3. The bisectors of two exterior angles of a triangle and of the interior angle at the third vertex meet in a point.



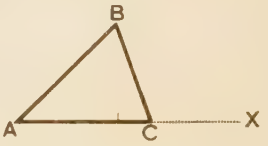
Ex. 4. The line through the vertex of an isosceles triangle, parallel to the base, bisects the exterior angle.

Ex. 5. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

Proof: $\angle DCB = 2\angle A$ (?) $= 2\angle DCR$ (?).

Proposition XXXIV. Theorem

104. The sum of the angles of any triangle is two right angles; that is, 180°.



Given: $\triangle ABC$.

To Prove:

$$\angle A + \angle B + \angle ACB = 2 \text{ rt. } \measuredangle = 180°.$$

Proof: Prolong AC to X, making the ext. $\angle BCX$.

$$\angle BCX + \angle ACB = 2 \text{ rt. } \measuredangle \quad (46).$$

But $$\angle BCX = \angle A + \angle B \quad (102).$$

$$\therefore \angle A + \angle B + \angle ACB = 2 \text{ rt. } \measuredangle = 180° \quad (\text{Ax. } 6).$$

Q.E.D.

105. Corollary. The sum of any two angles of a triangle is less than two right angles. (Ax. 5.)

106. Corollary. A triangle cannot have more than one right angle or more than one obtuse angle.

107. Corollary. Two angles of every triangle are acute.

108. Corollary. The acute angles of a right triangle are complementary.

Proof: Their sum = 1 rt. ∠. (104.)
Hence they are complementary. (19.)

109. Corollary. **Each angle of an equiangular triangle is 60°.**

110. Corollary. **If two right triangles have an acute angle of one equal to an acute angle of the other, the remaining acute angles are equal.** (48.)

111. Corollary. **If two triangles have two angles of the one equal to two angles of the other, the third angle of the first is equal to the third angle of the second.** (104.)

112. Corollary. **Two triangles are congruent if a side and any two angles of the one are equal respectively to a homologous side and the two homologous angles of the other.**

Proof: The third ∠ of one △ = third ∠ of other △ (111).
∴ the △ are ≅ (76).

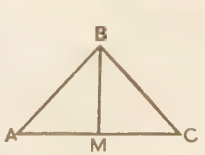
113. Corollary. **Two right triangles are congruent if a leg and the opposite acute angle of one are equal respectively to a leg and the opposite acute angle of the other.** (112.)

Ex. 1. If two angles of a triangle contain 50° and 100°, how many degrees are there in the other angle?

Ex. 2. Prove that if one angle of a triangle equals the sum of the other two angles, the triangle is a right triangle.

Ex. 3. How many degrees are there in each angle of an isosceles right triangle?

Ex. 4. The altitude of an isosceles right triangle upon the hypotenuse divides the triangle into two right triangles, each of whose acute angles is equal to 45°.



Ex 5. In a right triangle, if one acute angle is 47°, what is the other?

Ex. 6. In an isosceles triangle, if $\angle A = \angle B = 80°$, find $\angle C$.

Ex. 7. In $\triangle ABC$, if $\angle A = 25°$, $\angle B = 88°$, find $\angle C$ and the exterior angle at A.

Ex. 8. The vertex angle of an isosceles triangle is 44°. Find each base angle.

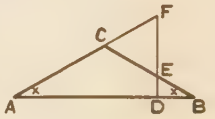
Ex. 9. If one acute angle of a right triangle is double the other, how many degrees are there in each?

Ex. 10. If one acute angle of a right triangle is five times the other, how many degrees are there in each?

Ex. 11. If any angle of an isosceles triangle is 60°, show that the triangle is equiangular.

Ex. 12. If the vertex angle of an isosceles triangle equals four times the sum of the base angles, find each angle.

Ex. 13. If the vertex angle of an isosceles triangle is twice the sum of the base angles, any line perpendicular to the base forms with the sides of the given triangle (one side to be produced) an equiangular triangle. [First find $\angle x$ and $\angle ACB$. Then use exterior $\measuredangle$ at D.]

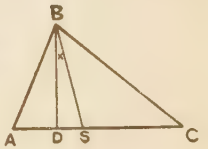


Ex. 14. The angles of a triangle are 44°, 62°, 74°. These are bisected by lines meeting at a point. Find the number of degrees in the three angles at this point.

Ex. 15. If two angles of a triangle are 80° and 55°, how many degrees are there in the angle formed by their bisectors?

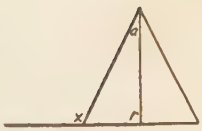
Ex. 16. The vertex angle of an isosceles triangle is one third of either exterior angle at the extremities of the base. Find each angle of the triangle.

Ex. 17. If two angles of a triangle are 30° and 40°, how many degrees are there in the angle formed by the bisector of the third angle and the altitude from the same vertex? **Solution:** $\angle x = \angle ABS - \angle ABD = \frac{1}{2}\angle ABC$ − comp. of $\angle A$.



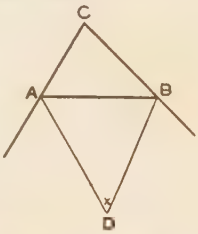
Ex. 18. The angle between the altitude of a triangle and the bisector of the angle at the same vertex equals half the difference of the other angles of the triangle.

Proof.
$$\angle x = ABS - \angle ABD$$
$$= \tfrac{1}{2}(180° - \angle A - \angle C) - (90° - \angle A) \qquad \textbf{(Ax. 6).}$$
$$= \text{etc.}$$



Ex. 19. The exterior angle at the base of an isosceles triangle equals half the vertex angle plus 90°.

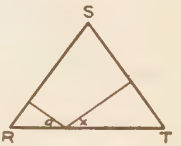
Ex. 20. If in $\triangle ABC$, $\angle BAC = 80°$, $\angle ABC = 30°$, find the angle formed by the bisectors of the exterior angles at A and B.



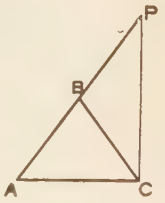
Ex. 21. The angle formed by the bisectors of two exterior angles of a triangle equals half the sum of the interior angles at the same vertices.

Ex. 22. If, in triangle ABC, the bisectors of the interior angle at B and the exterior angle at C meet at D, the angle BAC equals twice the angle BDC.

Ex. 23. The angle between the bisectors of two angles of a triangle equals half the third angle, plus a right angle.



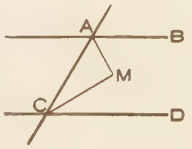
Ex. 24. If from any point in the base of an isosceles triangle perpendiculars to the equal sides are drawn, they make equal angles with the base.



Ex. 25. If one of the legs of an isosceles triangle is produced through the vertex its own length, and the extremity is joined to the nearer end of the base, this line is perpendicular to the base.

Ex. 26. If the middle point of one side of a triangle is equally distant from the three vertices, the triangle is a right triangle.

Ex. 27. If the points at which the bisectors of the equal angles of an isosceles triangle meet the opposite sides, are joined by a line, it is parallel to the base.



Ex. 28. The bisectors of two interior angles on the same side of a transversal cutting two parallels meet at right angles.

Proof: $\angle BAC + \angle ACD = 180°$ (?)

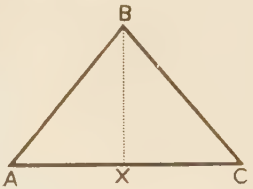
$\therefore \frac{1}{2}\angle BAC + \frac{1}{2}\angle ACD = 90°$ (Ax. 3).

That is, $\angle MAC + \angle MCA = 90°$ (Ax. 6).

$\therefore \angle M = 90°$ (104)

Proposition XXXV. Theorem

114. If two angles of a triangle are equal, the triangle is isosceles. [Converse of 55.]



Given: $\triangle ABC$; $\angle A = \angle C$.

To Prove: $AB = BC$.

Proof: Suppose BX drawn $\perp$ to AC.

In the rt. $\triangle$ ABX and CBX, $BX = BX$ (Iden.).

$\angle A = \angle C$ (Hyp.).

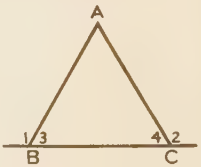
$\therefore \triangle ABX$ is congruent to $\triangle CBX$ (113).

$\therefore AB = BC$ (27).

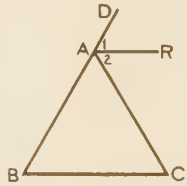
Q.E.D.

115. Corollary. An equiangular triangle is equilateral.

Ex. 1. If the exterior angles of a triangle are equal, the triangle is isosceles.

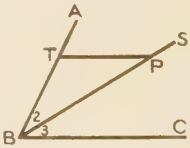


Ex. 2. A line parallel to the base of an isosceles triangle, meeting the equal sides, forms another isosceles triangle. Prove this for all three cases, whether it meets the equal sides, or those sides prolonged.



Ex. 3. If a line through the vertex of a triangle bisects the exterior angle and is parallel to the base, the triangle is isosceles.

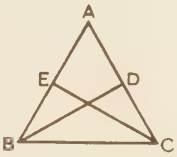
Ex. 4. If from any point in the bisector of an angle a line is drawn parallel to either side of the angle, an isosceles triangle is formed.



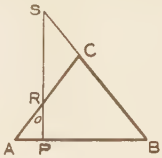
Ex. 5. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle.

Ex. 6. If from any point in the base of an isosceles triangle a line is drawn parallel to one of the equal sides and meeting the other side, an isosceles triangle is formed.

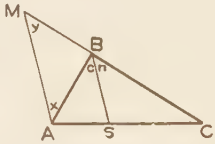
Ex. 7. If two altitudes of a triangle are equal, the triangle is isosceles. (Prove two ways.)



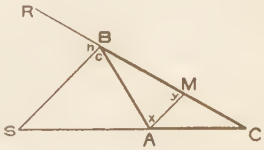
Ex. 8. If a perpendicular is erected at any point in the base of an isosceles triangle, meeting one leg, and the other leg produced, another isosceles triangle is formed.



Ex. 9. If ABC is a triangle, BS is the bisector of $\angle ABC$, and AM is parallel to BS, meeting BC produced, at M, the triangle ABM is isosceles.



Ex. 10. If a line is drawn perpendicular to the bisector of an angle and intersecting the sides, an isosceles triangle is formed.



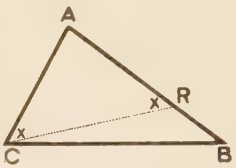
Ex. 11. If ABC is a triangle and BS is the bisector of exterior $\angle ABR$ and AM is $\parallel$ to BS, meeting BC at M, $\triangle ABM$ is isosceles.

EUCLID

Historical Note. Euclid of Alexandria lived during the third century B.C.; but little is known of his parentage, teachers, residence, or career. He was probably a contemporary of Eratosthenes and Archimedes. He wrote "The Elements," the most complete treatise on geometry that appeared before modern times, and inasmuch as this supplanted all others he must have made a great advance over the work of his predecessors. He was both a compiler and a discoverer. When a king of Egypt asked him about the possibility of mastering geometry with ease, Euclid is said to have replied, "There is no royal road to geometry."

Proposition XXXVI. Theorem

116. If two sides of a triangle are unequal, the angle opposite the longer side is greater than the angle opposite the shorter side.



Given: $\triangle ABC$; $AB > AC$.

To Prove: $\angle ACB > \angle B$.

Proof: On AB take $AR =$ to AC.

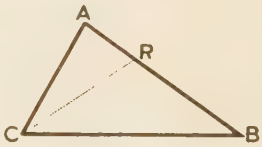
Draw CR and let $\angle ARC = x$.

	$\angle ARC$ is an ext. $\angle$ of $\triangle CBR$	(Def. 101).
	$\therefore \angle x > \angle B$	(103).
But	$\angle ACR = \angle x$	(55).
Substituting,	$\angle ACR > \angle B$	(Ax. 6).
Again,	$\angle ACB > \angle ACR$	(Ax. 5).
	$\therefore \angle ACB > \angle B$	(Ax. 11).

Q.E.D.

Proposition XXXVII. Theorem

117. If two angles of a triangle are unequal, the side opposite the greater angle is longer than the side opposite the less angle. [Converse.]



Given: $\triangle ABC$; $\angle ACB > \angle B$.

To Prove: $AB > AC$.

Proof: In $\angle ACB$, suppose $\angle BCR$ constructed $=$ to $\angle B$.

Then	$CR = BR$	(114).
Also	$AR + CR > AC$	(Ax. 12).
Substituting,	$AR + BR > AC$	(Ax. 6).
That is,	$AB > AC$.	Q.E.D.

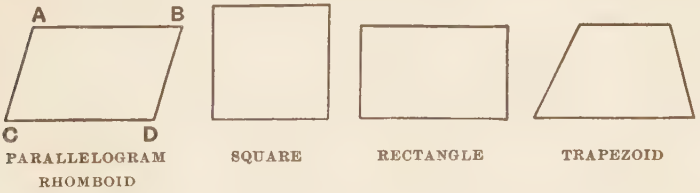
118. Corollary. The hypotenuse is the longest side of a right triangle. (117.)

QUADRILATERALS

119. A **quadrilateral** is a portion of a plane bounded by four straight lines. These four lines are called the **sides.** The **vertices** of a quadrilateral are the four points at which the sides intersect. The **angles** of a quadrilateral are the four angles at the vertices. The **diagonal** of a rectilinear figure is a line joining two vertices, not in the same side.

120. A **trapezium** is a quadrilateral having no two sides parallel. A **trapezoid** is a quadrilateral having two and only two sides parallel. A **parallelogram** is a quadrilateral having its opposite sides parallel (▱).

NOTE. In the first figure below, angles A, D, or B, C are **opposite** angles, angles A, C, or B, D, or A, B, or C, D are **consecutive** angles.



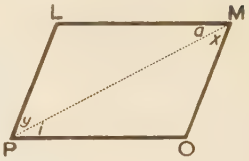
121. A **rectangle** is a parallelogram whose angles are right angles. A **rhomboid** is a parallelogram whose angles are not right angles.

122. A **square** is an equilateral rectangle. A **rhombus** is an equilateral rhomboid.

123. The side upon which a figure appears to stand is called its **base.** A trapezoid and all parallelograms have **two bases,** — the actual base and the side parallel to it. The non-parallel sides of a trapezoid are sometimes called the **legs.** An **isosceles trapezoid** is a trapezoid whose legs are equal. The **median** of a trapezoid is the line connecting the midpoints of the legs. The **altitude** of a trapezoid and of all parallelograms is the perpendicular distance between the bases.

Proposition XXXVIII. Theorem

124. The opposite sides of a parallelogram are equal.



Given: $\square\ LMOP$.

To Prove: $LM = PO$ and $LP = MO$.

Proof: Draw diagonal PM.

In $\triangle LMP$ and OMP, $PM = PM$ (Iden.).

$\angle a = \angle i$ (66).

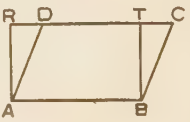
$\angle y = \angle x$ (?).

$\therefore \triangle LMP$ is congruent to $\triangle OMP$ (76).

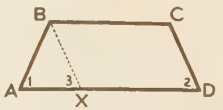
$\therefore LM = PO$ and $LP = MO$ (27).

Q.E.D.

Ex. 1. If two perpendiculars are drawn to the upper base of a parallelogram from the extremities of the lower base, two congruent right triangles are formed.



Ex. 2. The angles adjoining each base of an isosceles trapezoid are equal.

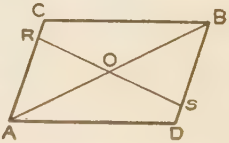


Ex. 3. If the angles at the base of a trapezoid are equal, the figure is isosceles.

Ex. 4. The diagonals of a rectangle are equal.

Ex. 5. If the diagonals of a parallelogram are equal, the figure is a rectangle.

Ex. 6. Any line terminated in a pair of opposite sides of a parallelogram and passing through the midpoint of a diagonal is bisected by this point.

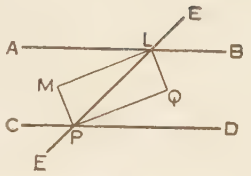


Ex. 7. If one angle of a parallelogram is a right angle, the figure is a rectangle.

Ex. 8. The bisectors of the angles of a trapezoid form a quadrilateral, two of whose angles are right angles.

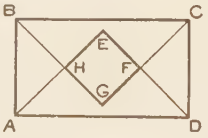
Ex. 9. The bisectors of the four interior angles formed by a transversal cutting two parallels form a rectangle.

[Prove each ∠ of $LMPQ$ a rt. ∠.]

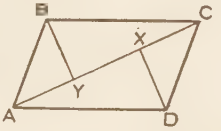


Ex. 10. The bisectors of the angles of a parallelogram form a rectangle.

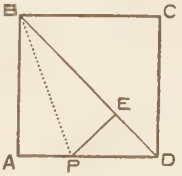
Ex. 11. The bisectors of the angles of a rectangle form a square. [In order to prove $EFGH$ equilateral, the △ AHB and CDF are proved congruent and isosceles; similarly △ BGC and AED.]



Ex. 12. The perpendiculars upon a diagonal of a parallelogram from the opposite vertices are equal.

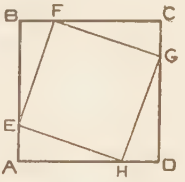


Ex. 13. If on diagonal BD, of square $ABCD$, BE is taken equal to a side of the square, and EP is drawn perpendicular to BD meeting AD at P, $AP = PE = ED$.



Ex. 14. If $ABCD$ is a square and E, F, G, H are points on the sides, such that $AE = BF = CG = DH$, $EFGH$ is a square.

[First, prove $EFGH$ equilateral; then one ∠ a rt. ∠.]



Ex. 15. The diagonals of an isosceles trapezoid are equal.

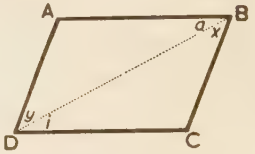
125. Corollary. **Parallel lines included between parallel lines are equal.** (124.)

126. Corollary. **The diagonal of a parallelogram divides it into two congruent triangles.**

127. Corollary. **The opposite angles of a parallelogram are equal.** (27.)

Proposition XXXIX. Theorem

128. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram. [Converse of 124.]



Given: Quadrilateral $ABCD$; $AB = DC$; $AD = BC$.

To Prove: $ABCD$ is a $\square$.

Proof: Draw diagonal BD.

In $\triangle ABD$ and CBD,	$BD = BD$	(Iden.).
	$AB = DC$ and $AD = BC$	(Hyp.).
	$\therefore \triangle ABD$ is congruent to $\triangle CBD$	(78).
	$\therefore \angle a = \angle i$	(27).
	$\therefore AB$ is $\parallel$ to DC	(70).
Also	$\angle y = \angle x$	(27).
	$\therefore AD$ is $\parallel$ to BC	(70).
Hence	$ABCD$ is a $\square$	(Def.). Q.E.D.

Proposition XL. Theorem

129. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

Given: Quadrilateral $ABCD$; $AB = DC$ and $AB \parallel$ to DC.

To Prove: $ABCD$ is a $\square$.

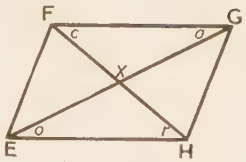
Proof: Draw diagonal BD.

In $\triangle ABD$ and CBD,	$BD = BD$	(Iden.).
	$AB = DC$	(Given).
	$\angle a = \angle i$	(66).
	$\therefore \triangle ABD$ is congruent to $\triangle CBD$	(52).
Hence	$\angle y = \angle x$	(27).
	$\therefore AD$ is $\parallel$ to BC	(70).
	$\therefore ABCD$ is a $\square$	(Def.). Q.E.D.

130. Corollary. Any pair of consecutive angles of a parallelogram are supplementary. (69.)

Proposition XLI. Theorem

131. The diagonals of a parallelogram bisect each other.



Given: $\square\ EFGH$; diagonals EG and FH intersecting at X.

To Prove: $FX = XH$ and $GX = XE$.

Proof: In $\triangle\ FXG$ and EXH, $FG = EH$ (124).

$\angle a = \angle o$ and $\angle c = \angle r$ (66).

$\therefore \triangle\ FXG$ is congruent to $\triangle\ EXH$ (76).

$\therefore FX = XH$ and $GX = XE$ (27). Q.E.D.

Proposition XLII. Theorem

132. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

Given: Quadrilateral $EFGH$, diagonals EG and FH, $FX = XH$, $EX = GX$.

To Prove: $EFGH$ is a $\square$.

Proof: In $\triangle\ FXG$ and EXH, $FX = XH$ and $EX = XG$ (Hyp.).

$\angle FXG = \angle EXH$ (?).

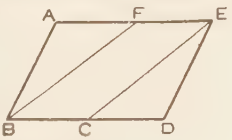
$\therefore \triangle\ FXG$ is congruent to $\triangle\ EXH$ (?).

$\therefore FG = EH$ and $\angle c = \angle r$ (?).

$\therefore FG$ is $\parallel$ to EH (70).

$\therefore EFGH$ is a $\square$ (129). Q.E.D.

Ex. 1. The lines joining a pair of opposite vertices of a parallelogram to the midpoints of the opposite sides are equal and parallel.

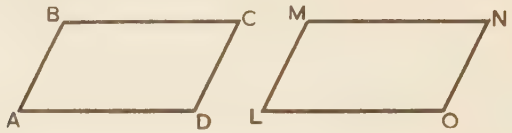


Ex. 2. If through the point of intersection of the diagonals of a parallelogram, two lines are drawn intersecting a pair of opposite sides (produced if necessary), the intercepts on these sides are equal.

Ex. 3. It is impossible to draw two straight lines from the ends of the base of a triangle terminating in the opposite sides, so that they shall bisect each other.

Proposition XLIII. Theorem

133. Two parallelograms are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.



Given: $\square$ AC and LN; $AB = LM$; $AD = LO$; $\angle A = \angle L$.

To Prove: The $\square$ are congruent.

Proof: Superpose $\square$ AC upon $\square$ LN, so that the equal $\measuredangle A$ and L coincide, AD falling along LO and AB along LM.

Point D coincides with point O [$AD = LO$ (Hyp.)].

Point B coincides with point M [$AB = LM$ (Hyp.)].

BC and MN are both $\parallel$ to LO (Def.)

$\therefore$ BC falls along MN (Ax. 13)

CD and NO are both $\parallel$ to LM (Def.).

$\therefore$ CD falls along NO (?).

Hence C falls exactly upon N (38).

$\therefore$ the figures coincide, and are congruent (26). Q.E.D.

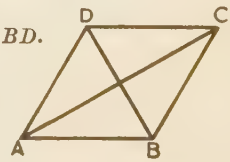
134. Corollary. Two rectangles are congruent if the base and the altitude of one are equal respectively to the base and the altitude of the other.

Proposition XLIV. Theorem

135. The diagonals of a rhombus (or of a square) are perpendicular to each other, bisect each other, and bisect the angles of the rhombus (or of the square).

Given: Rhombus $ABCD$; diagonals AC, BD.

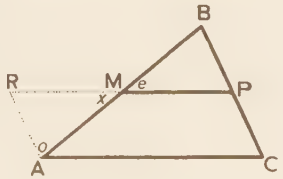
To Prove: $AC \perp$ to BD; AC and BD bisect each other; and they bisect $\measuredangle DAB$, ABC, etc.



Proof: Point A is equally distant from B and D (122).
Point C is equally distant from B and D (?).
$\therefore AC$ is $\perp$ to BD (83). Q.E.D.
Also AC and BD bisect each other (131). Q.E.D.
Also DB bisects $\angle ADC$ and $\angle ABC$ (85).
Similarly, AC bisects $\angle DAB$ and $\angle DCB$ (85).
Q.E.D.

Proposition XLV. Theorem

136. The line joining the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.



Given: $\triangle ABC$; M, the midpoint of AB; P, the midpoint of BC; line MP.

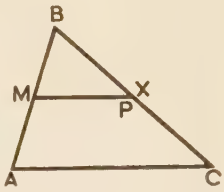
To Prove: $MP \parallel$ to AC;
and $MP = \frac{1}{2} AC$.

Proof: Suppose AR is drawn through A, $\parallel$ to BC and meeting MP produced, at R.

In $\triangle ARM$ and BPM, $AM = BM$ (Hyp.).
$\angle x = \angle e$ (51).
$\angle o = \angle B$ (66).
$\therefore \triangle ARM$ is congruent to $\triangle BPM$ (76).
$\therefore AR = BP$ (27).
But $BP = PC$ (Hyp.).
$\therefore AR = PC$ (Ax. 1).
$\therefore ACPR$ is a $\square$ (129).
Hence RP, or MP, is $\parallel$ to AC (Def.). Q.E.D.
Also $RP = AC$ (124).
But $MP = RM$ (27).
$\therefore MP$, the half of RP, $= \frac{1}{2} AC$ (Ax. 6).
Q.E.D.

Proposition XLVI. Theorem

137. The line bisecting one side of a triangle and parallel to a second side, bisects also the third side.

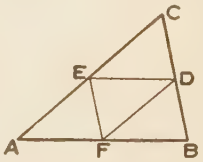


Given: $\triangle ABC$; MP bisecting AB and $\parallel$ to AC.

To Prove: MP bisects BC also.

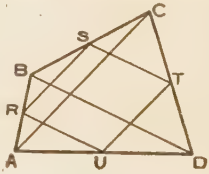
Proof: Suppose MX is drawn from M, the midpoint of AB to X, the midpoint of BC.

	MX is $\parallel$ to AC	(136).
But	MP is $\parallel$ to AC	(Hyp.).
	$\therefore MX$ and MP coincide	(Ax. 13).
	That is, MP bisects BC.	Q.E.D.



Ex. 1. The lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.

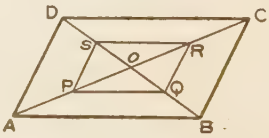
Ex. 2. The lines joining (in order) the midpoints of the sides of a quadrilateral form a parallelogram the sum of whose sides is equal to the sum of the diagonals of the quadrilateral.



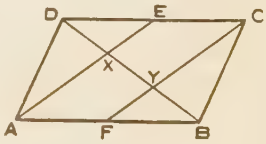
Ex. 3. The lines joining (in order) the midpoints of the sides of a rectangle form a rhombus. [Draw the diagonals.]

Ex. 4. If the four midpoints of the four halves of the diagonals of a parallelogram are joined in order, another parallelogram is formed.

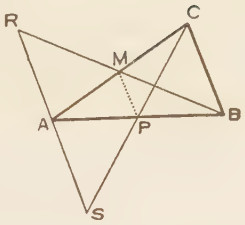
Prove this in four ways.



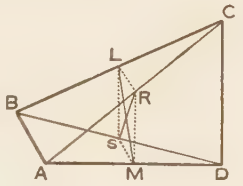
Ex. 5. If lines are drawn from a pair of opposite vertices of a parallelogram to the midpoints of a pair of opposite sides, they trisect the diagonal joining the other two vertices.



Ex. 6. If two medians are drawn from two vertices of a triangle and produced their own length beyond the opposite sides, and if these extremities are joined to the third vertex, these two lines are equal, and in the same straight line.



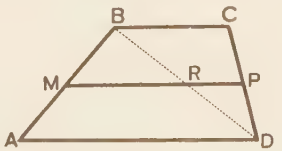
Ex. 7. The line joining the midpoints of one pair of opposite sides of a quadrilateral and the line joining the midpoints of the diagonals bisect each other.



Proposition XLVII. Theorem

138. The line bisecting one leg of a trapezoid and parallel to the base bisects the other leg, is the median, and is equal to half the sum of the bases.

Given: Trapezoid $ABCD$; M, the midpoint of AB; $MP \parallel$ to AD, meeting CD at P.



To Prove:

I. P is the midpoint of CD.

II. MP is the median.

III. $MP = \frac{1}{2}(AD + BC)$.

Proof: I. Draw diagonal BD, meeting MP at R.

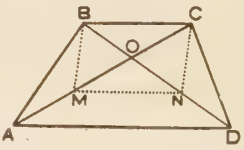
	MP is $\parallel$ to BC	(63).
In $\triangle ABD$,	MR bisects BD	(137).
In $\triangle BDC$,	RP bisects CD	(?).
That is,	P is the midpoint of CD.	Q.E.D.

II. MP is the median (Def. 123). Q.E.D.

III. In $\triangle ABD$,	$MR = \frac{1}{2} AD$	(136).
Also, in $\triangle BDC$,	$RP = \frac{1}{2} BC$	(?).
Adding,	$\therefore MP = \frac{1}{2}(AD + BC)$	(Ax. 2).

Q.E.D.

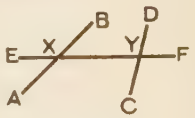
139. Corollary. The median of a trapezoid is parallel to the bases and equal to half their sum.



Ex. 1. In a trapezoid one of whose bases is double the other, the diagonals intersect at a point two thirds of the distance from each end of the longer base to the opposite vertex.

Proof: Take M, the midpoint of AO (136), etc.

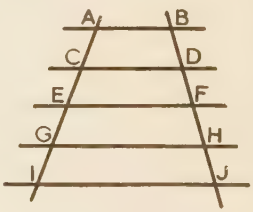
Ex. 2. If one angle of a triangle is double another, the line from the third vertex, making with the longer adjacent side an angle equal to the less given angle, divides the triangle into two isosceles triangles.



Note. The verb "*to intersect*" means merely "*to cut.*" In geometry, the verb "*to intercept*" means "*to include between.*" Thus the statement "AB and CD intercept XY on the line EF" really means, "AB and CD intersect EF and include XY, a part of EF, between them."

Proposition XLVIII. Theorem

140. Parallels intercepting equal parts on one transversal intercept equal parts on any transversal.



Given: ‖s AB, CD, EF, GH, IJ intercepting equal parts AC, CE, EG, GI, on the transversal AI, and cutting transversal BJ.

To Prove: $BD = DF = FH = HJ$.

Proof: The figure $ABFE$ is a trapezoid (?).

CD bisects AE and is ‖ to EF (Hyp.).

$\therefore$ D is midpoint of BF (?).

That is, $BD = DF$.

Similarly, $CDHG$ is a trapezoid and $DF = FH$.

Similarly, $FH = HJ$. $\therefore$ $BD = DF = FH = HJ$ (Ax. 1).

Q.E.D.

Proposition XLIX. Theorem

141. The midpoint of the hypotenuse of a right triangle is equally distant from the three vertices.

Given: Rt. $\triangle ABC$; M, the midpoint of the hypotenuse AB.

To Prove: $AM = CM = BM$.

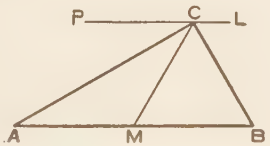
Proof: Suppose MX drawn $\parallel$ to BC, meeting AC at X.

	X is the midpoint of AC	(137).
	MX is $\perp$ to AC	(64).
	$\therefore AM = CM$	(80).
But	$AM = BM$	(Hyp.).
	$\therefore AM = CM = BM$	(Ax. 1).

Q.E.D.

Ex. 1. Any right triangle can be divided by one line into two isosceles triangles.

Ex. 2. If through the vertex of the right angle of a right triangle a line is drawn parallel to the hypotenuse, the legs of the right triangle bisect the angles formed by this parallel and the median drawn to the hypotenuse.

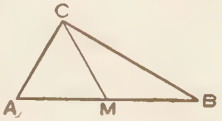


Ex. 3. If one leg of a trapezoid is perpendicular to the bases, the midpoint of the other leg is equally distant from the ends of the first leg. [Draw the median.]

Ex. 4. The median of a trapezoid bisects both the diagonals.

Ex. 5. The line joining the midpoints of the diagonals of a trapezoid is a part of the median, is parallel to the bases, and is equal to half their difference.

Ex. 6. If the median of a triangle is equal to half the side to which it is drawn, it is a right triangle.



Ex. 7. The line (prolonged if necessary) joining the midpoints of two sides of a triangle, bisects the altitude drawn to the third side.

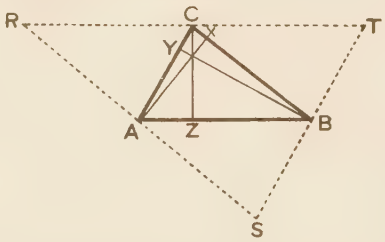
Ex 8. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

Proof: Use fig. of 141. Denote $\angle A$ by x,

then	$\angle B = 2x$ and $x = 30°$	(?).
	$AM = MC = BM$	(141).
	$\therefore \angle BCM = \angle B = 60°$	(55).
	$\therefore \angle BMC = 60°$	(104).
	$\therefore MB = BC$	(115).
	$\therefore AB$, or $2 \times MB$, $= 2 \times CB$	(Ax. 6).

Proposition L. Theorem

142. The perpendiculars from the vertices of a triangle to the opposite sides meet in a point.



Given: $\triangle ABC$, $AX \perp$ to BC, $BY \perp$ to AC, and $CZ \perp$ to AB.

To Prove: These three $\perp$s meet in a point.

Proof: Through A suppose RS drawn $\parallel$ to BC; through B, $TS \parallel$ to AC; through C, $RT \parallel$ to AB, forming $\triangle RST$.

	The figure $ABCR$ is a $\square$	(Const.).
	and $ABTC$ is a $\square$	(?).
	$RC = AB$ and $CT = AB$	(124).
	$\therefore RC = CT$	(Ax. 1).
Now	CZ is $\perp$ to RT	(64).
That is,	CZ is $\perp$ bisector of RT.	
Similarly,	AX is $\perp$ bisector of RS.	
And	BY is $\perp$ bisector of TS.	
	$\therefore$ in $\triangle RST$, AX, BY, CZ meet at a point	(100).

Q.E.D.

Proposition LI. Theorem

143. The three medians of a triangle meet in a point which is two thirds the distance from any vertex to the midpoint of the opposite side.

Given: $\triangle ABC$, medians AF, BD and CE, the latter two meeting at O. (Fig. 1.)

To Prove: $BO = \frac{2}{3} BD$; $CO = \frac{2}{3} CE$; $AO = \frac{2}{3} AF$ and that all three medians meet at O.

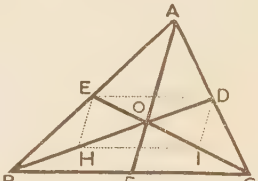


Fig. 1.

Proof: Suppose H is midpoint of BO and I is the midpoint of CO. Draw ED, DI, IH, HE.

In $\triangle ABC$, ED is $\parallel$ to BC and $= \frac{1}{2} BC$ (136).

In $\triangle OBC$, HI is $\parallel$ to BC and $= \frac{1}{2} BC$ (136).

$\therefore ED = HI$ (Ax. 1); and ED is $\parallel$ to HI (63).

$\therefore EDIH$ is a $\square$ (129).

$\therefore HO = OD$ and $IO = OE$ (131).

$\therefore BH = HO = OD$ and $CI = IO = OE$ (Ax. 1).

That is, $BO = \frac{2}{3} BD$ and $CO = \frac{2}{3} CE$.

Suppose AF meets BD at O'. (Fig. 2.)

Then $BO = \frac{2}{3} BD$ (Proved above).

And $BO' = \frac{2}{3} BD$ (Proved similarly).

$\therefore BO = BO'$ (Ax. 1).

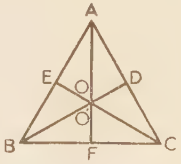
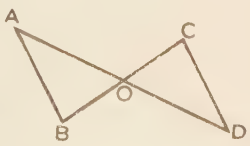


Fig. 2.

That is, O and O' are the same point, and the three medians meet at O, which is $\frac{2}{3}$ the distance from any vertex to the midpoint of the opposite side. Q.E.D.

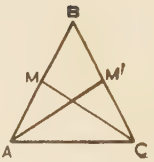
Ex. 1. If two lines (AB and CD) are equal and parallel, the lines connecting their opposite ends bisect each other.



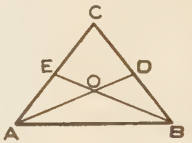
Ex. 2. If the bisector of one angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

Ex. 3. The median of an isosceles triangle is perpendicular to the base.

Ex. 4. The medians from the ends of the base of an isosceles triangle are equal.

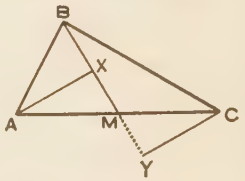


Ex. 5. If a triangle has two equal medians, it is isosceles.



Ex. 6. If ABC is an equilateral triangle and D, E, F are points on the sides, such that $AD = BE = CF$, triangle DEF is also equilateral.

Ex. 7. Any two vertices of a triangle are equally distant from the median from the third vertex.



Ex. 8. The lines bisecting two interior angles that a transversal makes with one of two parallels cut off equal segments on the other parallel from the point at which the transversal meets it. [The ▲ formed are isosceles.]

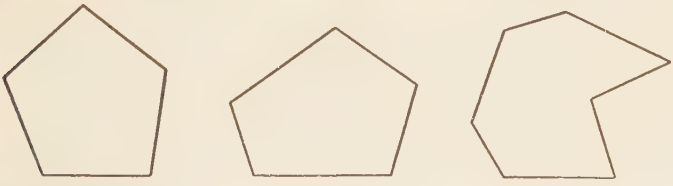
POLYGONS

144. A **polygon** is a portion of a plane bounded by straight lines. The lines are called the **sides.** The points of intersection of the sides are the **vertices.** The **angles** of a polygon are the angles at the vertices.

145. The **number** of sides of a polygon is the same as the number of its vertices or the number of its angles. An **exterior angle** of a polygon is an angle without the polygon, between one side of the polygon and another side prolonged.

146. An **equilateral polygon** has all its sides equal to one another. An **equiangular polygon** has all its angles equal to one another.

147. A **convex polygon** is a polygon no side of which if produced will enter the surface bounded by the sides of the polygon. A **concave polygon** is a polygon at least two sides of which if produced will enter the polygon.



EQUILATERAL CONVEX POLYGONS | EQUIANGULAR | CONCAVE, OR REËNTRANT

NOTE. A polygon may be equilateral and not be equiangular; or it may be equiangular and not be equilateral. The word "polygon" usually signifies *convex* figures.

148. Two polygons are **mutually equiangular** if for every angle of the one there is an equal angle in the other and similarly placed. Two polygons are **mutually equilateral** if for every side of the one there is an equal side in the other, and similarly placed.

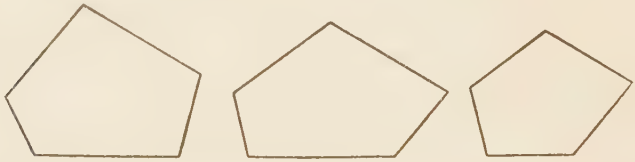
149. **Homologous angles** in two mutually equiangular polygons are the pairs of equal angles. **Homologous sides** in two polygons are the sides between two pairs of homologous angles.

150. Two **polygons** are **congruent** if they are mutually equiangular and their homologous sides are equal; or if they are composed of triangles, equal each to each and similarly placed. (Because in either case the polygons can be made to coincide.)

Ex. 1. If a quadrilateral has three equal sides, is it necessarily a parallelogram? a trapezoid?

Ex. 2. Two quadrilaterals are congruent if three sides and the two included angles of one are equal respectively to three corresponding sides and the two included angles of the other.

151. Two polygons may be mutually equiangular without being mutually equilateral; also, they may be mutually equilateral without being mutually equiangular — except in the case of triangles.

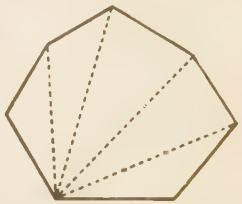


The first two figures are mutually equilateral, but *not* mutually equiangular. The last two figures are mutually equiangular, but *not* mutually equilateral.

152. A 3-sided polygon is a triangle.
A 4-sided polygon is a quadrilateral.
A 5-sided polygon is a pentagon.
A 6-sided polygon is a hexagon.
A 7-sided polygon is a heptagon.
An 8-sided polygon is an octagon.
A 10-sided polygon is a decagon.
A 12-sided polygon is a dodecagon.
A 15-sided polygon is a pentadecagon.
An n-sided polygon is called an n-gon.

Proposition LII. Theorem

153. The sum of the interior angles of an n-gon is equal to $(n-2)$ times 180°.



Given: A polygon having n sides.

To Prove: The sum of its interior $\measuredangle s = (n-2)\cdot 180°$.

Proof: If all possible diagonals are drawn from any vertex it is evident that there are formed $(n-2)$ triangles.

The sum of the $\measuredangle s$ of one $\triangle = 180°$ (104).

$\therefore$ the sum of the $\measuredangle$ of $(n-2)$ $\triangle = (n-2)\ 180°$ (Ax. 3).
The sum of $\measuredangle$ of the $\triangle$ = sum of $\measuredangle$ of n-gon (Ax. 4).
$\therefore$ the sum of $\measuredangle$ of the n-gon $= (n-2) \cdot 180°$ (Ax. 1). Q.E.D.

154. COROLLARY. **The sum of the interior angles of an n-gon is equal to $180° \cdot n - 360°$.**

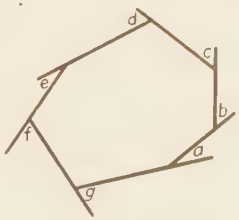
155. COROLLARY. **Each angle of an equiangular n-gon $= \frac{(n-2)\ 180°}{n}$.**

156. COROLLARY. **The sum of the angles of any quadrilateral is equal to four right angles.**

157. COROLLARY. **If three angles of a quadrilateral are right angles, the figure is a rectangle.**

PROPOSITION LIII. THEOREM

158. If the sides of a polygon are produced in order, one at each vertex, the sum of the exterior angles of the polygon equals four right angles, that is, 360°.



Given: A polygon with sides prolonged in succession forming the several exterior angles a, b, c, d, etc.

To Prove: $\angle a + \angle b + \angle c + \angle d + \text{etc.} = 4$ rt. $\measuredangle = 360°$.

Proof: Suppose at any point in the plane, lines are drawn parallel to the several sides of the given polygon, extending in the same direction, and forming $\measuredangle A$, B, C, D, etc.

Then $\angle A + \angle B + \angle C + \angle D + \angle E + \text{etc.} = 4$ rt. $\measuredangle$ (47).
But $\angle A = \angle a$, $\angle B = \angle b$, $\angle C = \angle c$, $\angle D = \angle d$, etc. (74).
Substituting,

$\angle a + \angle b + \angle c + \angle d + \text{etc.} = 4$ rt. $\measuredangle = 360°$ (Ax. 6).

Q.E.D.

159. Corollary. **Each exterior angle of an equiangular polygon is equal to $\frac{4 \text{ rt. } \angle\text{s}}{n}$, that is, to $\frac{360^\circ}{n}$.**

160. Corollary. **The sum of the exterior angles of a polygon is independent of the number of its sides.**

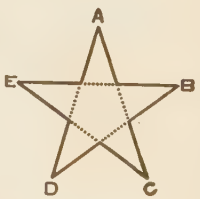
Ex. 1. How many degrees are there in the sum of all the angles of a pentagon? of a decagon? of a dodecagon?

Ex. 2. How many degrees are there in each angle of an equiangular hexagon? of an equiangular octagon?

Ex. 3. How many degrees are there in each exterior angle of an equiangular pentagon? of an equiangular hexagon? 16-gon?

Ex. 4. How many sides has the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 900°?

Ex. 5. The sum of the angles at the vertices of a five-pointed star (pentagram) is equal to two right angles.



Ex. 6. If two angles of a quadrilateral are supplementary, the other two are supplementary.

Ex. 7. If from any point within an angle perpendiculars to the sides are drawn, they include an angle which is the supplement of the given angle.

Ex. 8. If a quadrilateral has four equal sides, what kind is it?

Ex. 9. Can a trapezoid have three equal sides? a trapezium?

Ex. 10. A colt is tied, by a chain 30 ft. long, to the four corners of a lot 60 ft. square, on four successive days. Using a scale of $\frac{1}{8}$ in. to the foot, draw a diagram showing the area over which the colt grazes during these four days. Draw another diagram, using the same scale, for a chain 15 ft. long; 40 ft. long.

Ex. 11. A calf is tied by a chain 20 ft. long, to the four corners of a barn 40 ft. square, on four successive days and allowed to graze in the adjoining pasture. Using a scale of $\frac{1}{4}$ in. to the foot, draw a diagram showing the area grazed over at the end of the fourth day. Draw another diagram, using the same scale, for a chain 15 ft. long; 25 ft. long.

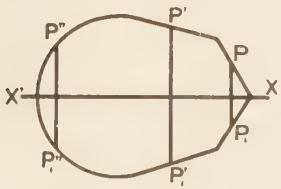
SYMMETRY

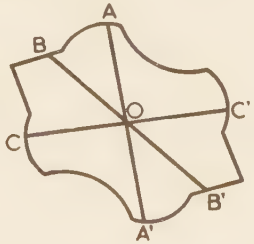
161. A figure is **symmetrical** with respect to a **line** if, by using that line as an axis, the part of the figure on one side of the line may be folded over, and will exactly coincide with the part on the other side. This line is an **axis of symmetry.**

162. A figure is **symmetrical** with respect to a **point** if this point bisects every line drawn through it and terminated (both ways) in the boundary of the figure.

This point is the **center of symmetry.**

163. It is evident that the axis of symmetry bisects at right angles every line joining two symmetrical points ; and that the center of symmetry bisects every line joining any pair of points symmetrical with respect to it.





Examples of symmetry are given in these figures.

First figure is symmetrical with respect to XX' as an axis. (Why?)

Second figure is symmetrical with respect to O as a center. (Why?)

P and P_1' are symmetrical with respect to XX' as an axis. (Why?)

A and A', B and B', etc. are symmetrical with respect to O as a center. (Why?) XX' is $\perp$ to $P'P_1'$ and bisects it. $AO = A'O$, $BO = B'O$, etc.

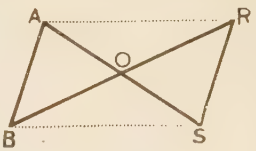
Ex. 1. Is the altitude of an isosceles triangle an axis of symmetry? the altitude of a scalene triangle?

Ex. 2. Is the diagonal of a rhomboid an axis of symmetry? of a square?

Ex. 3. Has any triangle a center of symmetry? a parallelogram?

Proposition LIV. Theorem

164. If two lines are symmetrical with respect to a center, they are equal and parallel.



Given: AB and RS symmetrical with respect to O, that is, every line through O, terminated in AB and RS, is bisected at O; AS and BR, two such lines.

To Prove: $AB = RS$ and $AB \parallel$ to RS.

Proof: Draw AR and BS. $AO = OS$ and $BO = OR$ (Hyp.).

$\therefore$ $ABSR$ is a $\square$ (132).

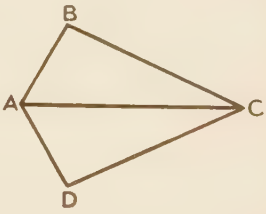
$\therefore$ $AB = RS$ (124).

Also AB is $\parallel$ to RS (Def. 120).

Q.E.D.

Proposition LV. Theorem

165. If a diagonal of a quadrilateral bisects two of its angles, this diagonal is an axis of symmetry.



Given: Quadrilateral $ABCD$: AC a diagonal bisecting $\angle BAD$ and $\angle BCD$.

To Prove: $ABCD$ symmetrical with respect to AC.

Proof: In $\triangle ABC$ and ADC, $AC = AC$ (?).

$\angle BAC = \angle DAC$ and $\angle BCA = \angle DCA$ (?).

$\therefore$ $\triangle ABC$ is congruent to $\triangle ADC$ (?).

$\therefore$ AC is an axis of symmetry (161).

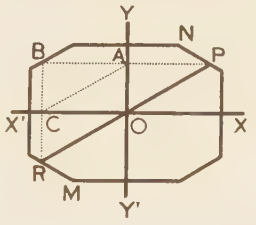
Q.E.D.

166. Corollary. The diagonal of a square or of a rhombus is an axis of symmetry.

Ex. The point of intersection of the diagonals of a parallelogram is a center of symmetry. Prove.

Proposition LVI. Theorem

167. If a figure is symmetrical with respect to two perpendicular axes, it is symmetrical with respect to their intersection as a center.



Given: Figure MN symmetrical with respect to the $\perp$ axes XX' and YY' which intersect at O.

To Prove: Figure MN is symmetrical with respect to O as a center.

Proof: Take any point P in the boundary. Draw $PB \perp$ to YY', intersecting YY' at A and meeting the boundary at B. Draw $BR \perp$ to XX', intersecting XX' at C and meeting the boundary at R. Draw AC, OP, OR.

[The demonstration is accomplished by proving POR a straight line, bisected at O.]

	PB is $\parallel$ to XX' and BR is $\parallel$ to YY'	(62).
	$\therefore ABCO$ is a $\square$	(Def.).
	$\therefore BC = AO$	(124).
But	$BC = CR$	(161).
	$\therefore AO = CR$	(Ax. 1).
Hence	$ACRO$ is a $\square$	(129).
	$\therefore RO = CA$	(124).
Also	RO is $\parallel$ to CA	(120).
Similarly,	$ACOP$ may be proved a $\square$.	
	$\therefore PO$ is $=$ to AC and $\parallel$ to AC.	
Hence	POR is a straight line	(Ax. 13).
And	$PO = RO$	(Ax. 1).

But P is *any* point in the boundary, so POR is *any* line through O.

$\therefore O$ is a center of symmetry (161).

Q.E.D.

Ex. Prove that no triangle can have a center of symmetry.

CONCERNING ORIGINAL EXERCISES

168. In the original work which this text contains, the pupil is expected to state the hypothesis and the conclusion of each theorem, and to apply them to an appropriate figure; also to give a complete and logical statement of the proof, with a reason for every statement.

In many of these exercises, suggestions are made and such assistance is given as experience has shown to be needed by average pupils. This is done in order to encourage definite accomplishment, which is one of the greatest incentives to further effort.

To apply the knowledge acquired from the preceding pages is now the student's task. His interest in this science will depend largely on the success of his efforts to prove originals.

The student should not draw a special figure for a general proposition. That is, if "triangle" is specified, he should draw a scalene and not an isosceles or a right triangle; and if "quadrilateral" is mentioned, he should draw a trapezium and not a parallelogram or a square.

SUMMARY. GENERAL DIRECTIONS FOR ATTACKING EXERCISES

169. A **triangle** is proved **isosceles** by showing that it contains two equal sides, or two equal angles.

170. A **triangle** is proved a **right** triangle by showing that one of its angles is a right angle, or that two of its angles are complementary, or that one of its angles is equal to the sum of the other two.

171. **Right triangles** are proved congruent by showing that they have:

(1) Hypotenuse and acute angle of one equal etc.
(2) Hypotenuse and leg of one equal etc.
(3) The legs of one equal etc.
(4) Leg and adjoining angle of one equal etc.
(5) Leg and opposite angle of one equal etc.

172. Oblique triangles are proved congruent by showing that they have :

(1) Two sides and the included angle of one equal etc.
(2) One side and the adjoining angles of one equal etc.
(3) Three sides of one equal etc.

173. Angles are proved equal by showing that they are :

(1) Equal to the same or to equal angles.
(2) Halves or doubles of equals.
(3) Vertical angles.
(4) Complements or supplements of equals.
(5) Homologous parts of congruent figures.
(6) Base angles of an isosceles triangle.
(7) Corresponding angles, alternate interior angles, etc., of parallels.
(8) Angles whose sides are respectively parallel or perpendicular.
(9) Third angles of triangles which have two angles of one equal etc.

174. Lines are proved equal by showing that they are :

(1) Equal to the same or to equal lines.
(2) Halves or doubles of equals.
(3) Distances to the ends of a line from any point in its perpendicular bisector.
(4) Homologous parts of congruent figures.
(5) Sides of an isosceles triangle.
(6) Distances to the sides of an angle from any point in its bisector.
(7) Opposite sides of a parallelogram.
(8) The parts of one diagonal of a parallelogram made by the other.

175. Two lines are proved **perpendicular** by showing that they:

(1) Make equal adjacent angles with each other.
(2) Are legs of a right triangle.
(3) Have two points in one, each equally distant from the ends of the other.

176. Two lines are proved **parallel** by:

(1) The customary angle relations of parallel lines.
(2) Showing that they are opposite sides of a parallelogram.
(3) Showing that they are parallel or perpendicular to a third line.

177. Two lines, or two angles, are proved **unequal** by the usual axioms and theorems pertaining to inequalities.

[See especially, Ax. 5; Ax. 12; 81, 86, 87, 88, III, 91, 92, 103, 116, 118.]

ORIGINAL EXERCISES

1. Parallel lines are everywhere equally distant.

Given: $\parallel_s$ AC and BD; AB and $CD \perp$ to AC.

To Prove: $AB = CD$.

2. If two lines in a plane are everywhere equally distant, they are parallel.

3. The bisectors of any two consecutive angles of a parallelogram meet at right angles.

4. The line drawn from any point in the base of an isosceles triangle to the opposite vertex is less than either leg.

5. If ABC is an equilateral triangle and each side is produced (in order) the same distance, so that $AD = BE = CF$, the triangle DEF is equilateral.

6. If $ABCD$ is a square and the sides are produced (in order) the same distance, so that $AE = BF = CG = DH$, the figure $EFGH$ is a square.

7. The two lines joining the midpoints of the opposite sides of a quadrilateral bisect each other. [Join the 4 midpoints (in order), etc.]

8. If two adjacent angles of a quadrilateral are right angles, the bisectors of the other angles are perpendicular to each other.

9. If two opposite angles of a quadrilateral are right angles, the bisectors of the other angles are parallel.

10. Two isosceles triangles are congruent, if:

(1) The base and one of the adjoining angles in the one are equal respectively to the base and one of the adjoining angles in the other.

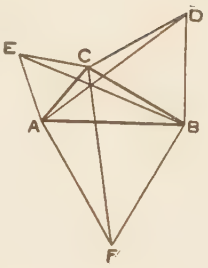
(2) A leg and one of the base angles in the one are equal respectively to a leg and one of the base angles in the other.

(3) The base and vertex angle in one are equal to the same in the other.

(4) A leg and vertex angle in one are equal to the same in the other.

(5) A leg and the base in one are equal to the same in the other.

11. If upon the three sides of any triangle equilateral triangles are constructed (externally) and a line is drawn from each vertex of the given triangle to the farthest vertex of the opposite equilateral triangle, these three lines are equal.



Proof: $\angle EAC = \angle BAF$ (?). Add to each of these $\angle CAB$.

$\therefore \angle EAB = \angle CAF$ (?).

Then prove $\triangle EAB$ and CAF congruent.

Similarly, $\triangle CAD$ is congruent to $\triangle CEB$. Etc.

12. The sum of the diagonals of any quadrilateral is less than the sum of the four sides, but greater than half that sum.

13. The difference between two sides of a triangle is less than the third side.

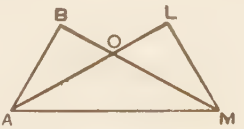
14. The bisectors of the exterior angles of a rectangle form a square.

15. The bisectors of the equal angles of an isosceles triangle (terminating in the equal sides) are equal.

16. The median to the base of an isosceles triangle bisects the vertex angle.

17. The perpendiculars to the legs of an isosceles triangle from the midpoint of the base are equal.

18. State and prove the converse of Ex. 17.



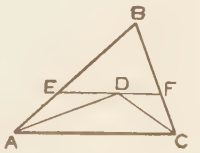
19. If $AB = LM$ and $AL = BM$, $\angle B = \angle L$ and $\angle BAO = \angle OML$ and $BO = OL$.

20. The bisectors of a pair of corresponding angles are parallel.

21. If two lines are cut by a transversal and the exterior angles on the same side of the transversal are supplementary, the lines are parallel.

22. The bisectors of a pair of vertical angles are in the same straight line.

23. The midpoint of a diagonal of a parallelogram is a center of symmetry.

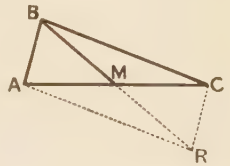


24. If the base angles of a triangle are bisected and through the intersection of the bisectors a line is drawn parallel to the base and terminating in the sides, this line is equal to the sum of the parts of the sides it meets, between it and the base.

25. In two congruent triangles, homologous medians are equal; homologous altitudes are equal; homologous bisectors are equal.

26. If two parallel lines are cut by a transversal, the two exterior angles on the same side of the transversal are supplementary.

27. If from a point a perpendicular is drawn to each of two parallels, they are in the same line. [Draw a third ∥ through the point.]



28. The median to one side of a triangle is less than half the sum of the other two sides.

Proof: Produce median BM to R, so that $MR = BM$, draw AR and CR. Fig. is $\square$. (?). Etc.

29. The sum of the medians of a triangle is less than the sum of the sides of the triangle.

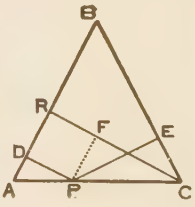
30. If the diagonals of a trapezoid are equal, it is isosceles.

[Draw DR and $CS \perp$ to AB; and prove rt. $\triangle ACS$ and BDR congruent, to get $\angle x = \angle x$.]

31. If a perpendicular is drawn from each vertex of a parallelogram to any line outside the parallelogram, the sum of the ⊥s from one pair of opposite vertices equals the sum of the ⊥s from the other pair.

32. The sum of the perpendiculars to the legs of an isosceles triangle from any point in the base equals the altitude upon one of the legs. (That is, the sum of the perpendiculars from any point in the base of an isosceles triangle to the equal sides remains a uniform length for every point of the base.)

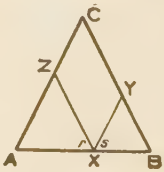
[Prove $PE = CF$.]



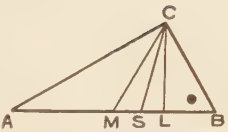
33. The sum of the three perpendiculars drawn from any point within an equilateral triangle, to the three sides, remains a uniform length for all positions of the point.

[Draw a line through this point ∥ to one side; draw the altitude of the $\triangle \perp$ to this line and side; prove the sum of the three ⊥s equals this altitude and hence equals a constant.]

34. If from any point in the base of an isosceles triangle parallels to the equal sides are drawn, the sum of the sides of the parallelogram formed is equal to the sum of the legs of the triangle.



35. The bisector of the right angle of a right triangle is also the bisector of the angle formed by the median and the altitude drawn from the same vertex.

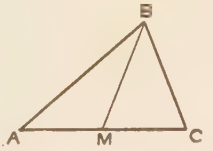


To Prove: $\angle MCS = \angle LCS$.

Proof: $\angle ACS = \angle BCS$ (?); $\angle ACM = \angle BCL$ (?). Now use Ax. 2.

36. If the vertex angle of an isosceles triangle is equal to the sum of the base angles, any line perpendicular to the base forms with the sides of the given triangle (one side to be produced) three isosceles right triangles.

37. If two sides of a triangle are unequal and the median to the third side is drawn, the angles formed with the base are unequal.



38. State and prove the converse of Ex. 37.

39. If the opposite sides of a hexagon are equal and parallel, the three diagonals drawn between opposite vertices meet in a point.

40. In triangle ABC, AD is perpendicular to BC, meeting it at D; E is the midpoint of AB, and F of AC; the angle EDF is equal to the angle EAF.

41. If the diagonals of a quadrilateral are equal, and also one pair of opposite sides, two of the four triangles into which the quadrilateral is divided by the diagonals are isosceles.

42. If angle A of triangle ABC equals three times angle B, there can be drawn a line AD meeting BC in D, such that the triangles ABD and ACD are isosceles.

43. If E is the midpoint of side BC of parallelogram $ABCD$, AE and BD meet at a point two thirds the distance from A to E and from D to B.

44. If in triangle ABC, in which AB is not equal to AC, AC' is taken on AB (produced if necessary) equal to AC, and AB' is taken on AC (produced if necessary) equal to AB, and $B'C'$ is drawn meeting BC at D, then AD bisects angle BAC.

Proof: $\triangle ABC$ is congruent to $\triangle AB'C'$ (?) (52). $\therefore$ their homologous parts are equal. Thus prove that $\triangle BC'D$ is congruent to $\triangle B'CD$. Etc.

45. If a diagonal of a parallelogram bisects one angle, it also bisects the opposite angle.

46. If a diagonal of a parallelogram bisects one angle, the figure is equilateral.

47. Any line drawn through the point of intersection of the diagonals of a parallelogram divides the figure into two congruent quadrilaterals.

48. If AR bisects angle A of triangle ABC and AT bisects the exterior angle at A, any line parallel to AB, having its extremities in AR and AT, is bisected by AC.

49. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

50. If, in isosceles triangle XYZ, AD is drawn from A, the midpoint of YZ, perpendicular to the base XZ, $DZ = \frac{1}{4}\ XZ$. [Draw alt. from Y.]

51. If ABC is an equilateral triangle, if the bisectors of angles B and C meet at D, if DE is drawn parallel to AB meeting AC at E, and DF, parallel to BC meeting AC at F, then $AE = ED = EF = DF = CF$.

52. If A is any point in RS of triangle RST, and B is the midpoint of RA, C the midpoint of AS, D the midpoint of ST, and E the midpoint of TR, then $BCDE$ is a parallelogram.

53. If lines are drawn from any vertex of a parallelogram to the midpoints of the two opposite sides, they divide the diagonal which they intersect into three equal parts.

Proof: Draw the other diagonal.

54. If the interior and exterior angles at two vertices of a triangle are bisected, a quadrilateral is formed, having two of its angles right angles and the other two supplementary.

55. The four bisectors of the angles of a quadrilateral form a second quadrilateral whose opposite angles are supplementary.

Proof: Extend a pair of opposite sides of the given quadrilateral to meet at X. Bisect the base angles of the new $\triangle$ formed, meeting at O. Then show that $\angle O$ equals one of the $\measuredangle$ between the given bisectors, and $\angle O$ is supplementary to the angle opposite.

BOOK II

THE CIRCLE

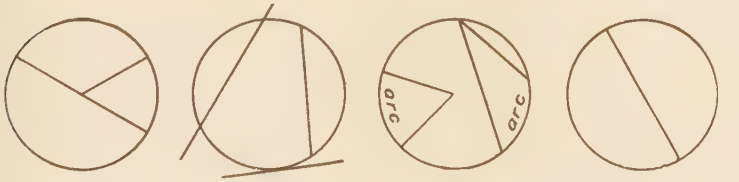
178. A **curved line** is a line no part of which is straight.

179. A **circle** is a plane curve all points of which are equally distant from a point in the plane, called the **center**.

180. The length of the circle is called the **circumference**.

181. A **radius** is a straight line drawn from the center to any point of the circle.

A **diameter** is a straight line that contains the center, and the extremities of which are in the circle.



CIRCLE
RADIUS
DIAMETER

SECANT CHORD
TANGENT
POINT OF CONTACT

CENTRAL ANGLE
INSCRIBED ANGLE
ARC

SEMI-
CIRCUMFERENCES
SEMICIRCLES

A **secant** is a straight line cutting the circle in two points.

A **chord** is a straight line the extremities of which are in the circle.

A **tangent** is a straight line which touches the circle at only one point, and does not cut it, however far it may be extended. The point at which the line touches the circle is called the **point of contact** or the **point of tangency**.

A **common tangent** to two circles is a line tangent to both of them.

182. A **central angle** is an angle formed by two radii.

An **inscribed angle** is an angle whose vertex is on the circle and whose sides are chords.

183. An **arc** is any part of a circle.

A **semicircle** is an arc equal to half a circle.

A **quadrant** is an arc equal to one fourth of a circle.

Equal circles are circles having equal radii.

Concentric circles are circles having the same center.

CIRCLE O | SECTOR SEGMENT | CIRCLES INTERNALLY TANGENT | CIRCLES EXTERNALLY TANGENT

NOTE. A **circle** is **named** either by its center or by three of its points as "the ⊙ *O*," or "the ⊙ *ABC*."

184. A **sector** is the figure bounded by two radii and their included arc.

A **segment** is the figure bounded by an arc and its chord.

185. Two **circles** are **tangent** to each other if they are tangent to the same line at the same point. Circles may be tangent to each other **internally**, if the one is within the other, or **externally**, if each is without the other.

186. POSTULATE. **A circle can be described about any given point as center and with any given line as radius.**

Subtend is used in the sense of "to cut off." A chord **subtends** an arc. Hence an arc is **subtended by** a chord.

An **angle** is said to **intercept** the arc between its sides. Hence an arc is **intercepted by** an angle.

The **hypothesis** is contained in what constitutes the subject of the principal verb of the theorem.

PRELIMINARY THEOREMS

187. THEOREM. **All radii of the same circle are equal.** (179.)

188. THEOREM. **All radii of equal circles are equal.** (183.)

189. THEOREM. **The diameter of a circle equals twice the radius.**

190. THEOREM. **All diameters of the same or of equal circles are equal.** (Ax. 3.)

191. THEOREM. **The diameter of a circle bisects the circle.**

Given: Any ⊙ and a diameter.

To Prove: The diameter bisects the circle.

Proof: Suppose one segment folded over upon the other segment, using the diameter as an axis. If the arcs do not coincide, there are points of the circle unequally distant from the center. But this is impossible (179).

∴ the arcs coincide and are equal (26).

Q.E.D.

192. THEOREM. **With a given point as center and a given line as radius, it is possible to describe only one circle.** (179.)

That is, a circle is determined if its center and radius are fixed.

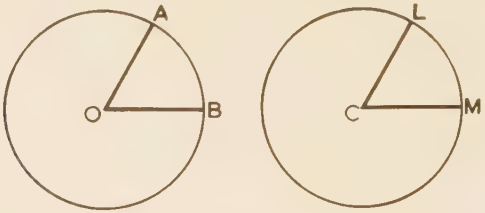
Historical Note. Archimedes was born at Syracuse, Sicily, during the third century B.C. He is regarded as the greatest mathematician of antiquity, and probably of all time, save only that modern wizard, Sir Isaac Newton. He was educated in Egypt and won the respect and admiration of the king, Hiero, for his exceptional genius in the construction of mechanical devices and mathematical formulas. These included the measurement of the circle (circumference and area), the cone, the cylinder, and the sphere. To him is given credit also for the discovery of specific gravity. Cicero relates his discovery of the tomb of Archimedes over a century after his burial in Syracuse.

ARCHIMEDES

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

193. In the same circle (or in equal circles) equal central angles intercept equal arcs.



Given: $\odot O = \odot C$; $\angle O = \angle C$.

To Prove: Arc AB = arc LM.

Proof: Superpose $\odot O$ upon the equal $\odot C$, making $\angle O$ coincide with its equal, $\angle C$. Point A falls on L, and point B on M (188).

Arc AB coincides with arc LM (179).

$\therefore$ arc AB = arc LM (26).

Q.E.D.

Proposition II. Theorem

194. In the same circle (or in equal circles) equal arcs are intercepted by equal central angles. [Converse.]

Given: $\odot O = \odot C$; arc AB = arc LM.

To Prove: $\angle O = \angle C$.

Proof: Superpose $\odot O$ upon the equal $\odot C$, making the centers coincide. Point A falls on point L. Then arc AB coincides with arc LM and point B falls on point M. (Because the arcs are equal.)

$\therefore$ OA coincides with CL, and OB with CM (39).

$\therefore \angle O = \angle C$ (26).

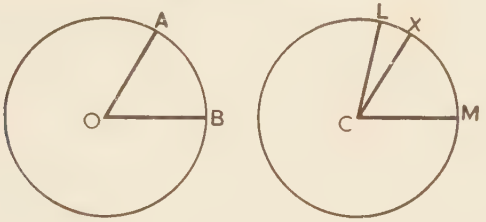
Q.E.D.

Proposition III. Theorem

195. In the same circle (or in equal circles) :

I. If two central angles are unequal, the greater angle intercepts the greater arc.

II. If two arcs are unequal, the greater arc is intercepted by the greater central angle. [Converse.]



I. **Given:** $\odot O = \odot C$; $\angle LCM > \angle O$.

To Prove: Arc $LM >$ arc AB.

Proof: Superpose $\odot O$ upon $\odot C$, making sector AOB fall in position of sector XCM, OB coinciding with CM.

CX is within the angle LCM. (Because $\angle LCM > \angle O$.)

Arc AB falls upon LM, in the position XM (179).

$\therefore$ arc $LM >$ arc XM (Ax. 5).

That is, arc $LM >$ arc AB. Q.E.D.

II. **Given:** (?).

To Prove: $\angle LCM > \angle O$.

Proof: The pupil may employ either superposition, as in I, or the method of exclusion, as in 92.

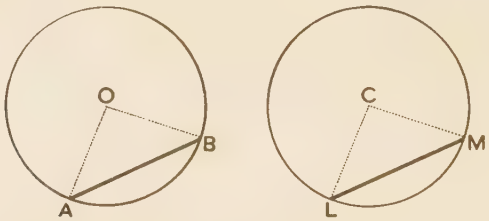
Note. Unless otherwise specified, the arc of a chord always refers to the *lesser* of the two arcs. If *two* arcs (in the same or equal circles) are concerned, it is understood either that each is less than a semicircle, or each is greater.

Ex. 1. Are equal circles also congruent? Why?

Ex. 2. Is there a geometrical figure that is both sector and segment?

Proposition IV. Theorem

196. In the same circle (or in equal circles) equal chords subtend equal arcs.



Given: $\odot O = \odot C$; chord AB = chord LM.

To Prove: Arc AB = arc LM.

Proof: Draw the several radii to the ends of the chords.

In $\triangle OAB$ and CLM, $OA = CL$ and $OB = CM$ (188).

Chord AB = chord LM (Hyp.).

$\therefore \triangle OAB$ is congruent to $\triangle CLM$ (?).

Hence $\angle O = \angle C$ (?).

$\therefore$ arc AB = arc LM (193).

Q.E.D.

Proposition V. Theorem

197. In the same circle (or in equal circles) equal arcs are subtended by equal chords. [Converse.]

Given: $\odot O = \odot C$; arc AB = arc LM.

To Prove: Chord AB = chord LM.

Proof: Draw the several radii to the ends of the chords.

In $\triangle OAB$ and CLM, $OA = CL$ (188).

$OB = CM$ (?).

$\angle O = \angle C$ (194).

$\therefore \triangle AOB$ is congruent to $\triangle CLM$ (?).

$\therefore$ chord AB = chord LM (?).

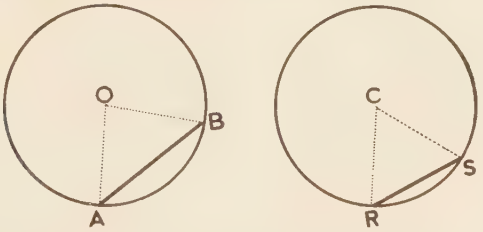
Q.E.D.

Proposition VI. Theorem

198. In the same circle (or in equal circles):

I. **If two chords are unequal, the greater chord subtends the greater arc.**

II. **If two arcs are unequal, the greater arc is subtended by the greater chord.** [Converse.]



I. **Given:** $\odot O = \odot C$; chord $AB >$ chord RS.

To Prove: Arc $AB >$ arc RS.

Proof: Draw the several radii to the ends of the chords.

In $\triangle AOB$ and RCS,

$AO = RC$ and $BO = SC$	(?).
Chord $AB >$ chord RS	(Hyp.).
$\therefore \angle O > \angle C$	(92).
$\therefore$ arc $AB >$ arc RS	(195, I).

Q.E.D.

II. **Given:** $\odot O = \odot C$; arc $AB >$ arc RS.

To Prove: Chord $AB >$ chord RS.

Proof: Draw the several radii.

In $\triangle AOB$ and RCS,

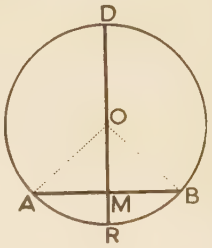
$AO = RC$ and $BO = SC$	(?).
But $\angle O > \angle C$	(195, II).
$\therefore$ chord $AB >$ chord RS	(91).

Q.E.D.

Ex. Can either part of Proposition VI be proved by the method of exclusion? Can Proposition IV or V be proved by that method?

Proposition VII. Theorem

199. The diameter perpendicular to a chord bisects the chord and both the subtended arcs.



Given: Diameter $DR \perp$ to chord AB in $\odot O$.

To Prove:

I. $AM = MB$;

II. Arc AR = arc RB, arc AD = arc DB.

Proof: Draw radii to ends of the chord.

I. In rt. $\triangle OAM$ and OBM,

$OA = OB$	(?).
$OM = OM$	(?).
$\triangle OAM$ is congruent to $\triangle OBM$	(84).
$\therefore AM = BM$	(?).
	Q.E.D.

II.	$\angle AOM = \angle BOM$	(27).
	$\therefore$ arc AR = arc RB	(193).
Also	$\angle AOD = \angle BOD$	(49).
	$\therefore$ arc AD = arc DB	(193).
		Q.E.D.

200. Corollary. The line from the center of a circle perpendicular to a chord bisects the chord.

201. Corollary. The perpendicular bisector of a chord passes through the center of the circle.

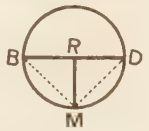
Proof: The center is equally distant from the extremities of the chord (187).

$\therefore$ the center is in the $\perp$ bisector of the chord (82).

Ex. 1. The perpendicular bisectors of all chords in a circle pass through a common point.

Ex. 2. A diameter bisecting a chord is perpendicular to the chord and bisects the subtended arcs.

Ex. 3. A diameter bisecting an arc is the perpendicular bisector of the chord of the arc.



Ex. 4. A line bisecting a chord and its arc is the perpendicular bisector of the chord.

Ex. 5. If a circle is described on the hypotenuse of a right triangle as diameter, it passes through the vertex of the right angle (141).

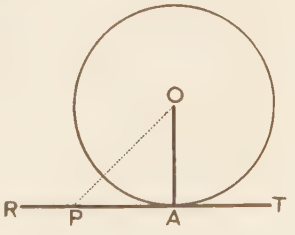
Ex. 6. If any number of parallel chords are drawn in a circle, their midpoints all lie on the same straight line.

Ex. 7. If two perpendicular diameters of a circle are drawn and their extremities are joined in order, these chords form a square.

Ex. 8. If any two diameters of a circle are drawn and their extremities are joined in order, the figure is a parallelogram.

Proposition VIII. Theorem

202. The line perpendicular to a radius at its extremity is tangent to the circle.



Given: Radius OA of $\odot O$, and $RT \perp$ to OA at A.

To Prove: RT tangent to the circle.

Proof: Take any point P in RT (except A) and draw OP.

$OP > OA$ (87).

$\therefore$ P lies without the $\odot$ (Because $OP >$ radius).

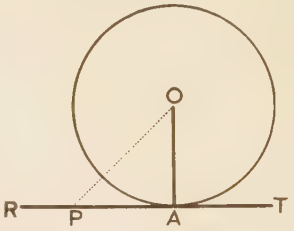
That is, every point (except A) in RT is without the $\odot$.

$\therefore$ RT is a tangent to the $\odot O$ (Def.).

Q.E.D.

Proposition IX. Theorem

203. If a line is tangent to a circle, the radius drawn to the point of contact is perpendicular to the tangent. [Converse.]



Given: RT tangent to $\odot O$ at A; radius OA.

To Prove: $OA \perp$ to RT.

Proof: Every point (except A) in RT is without the $\odot$ (181).

$\therefore$ a line from O to any point in RT (except A) is $> OA$. (Because it is $>$ a radius.)

That is, OA is the shortest line from O to RT.

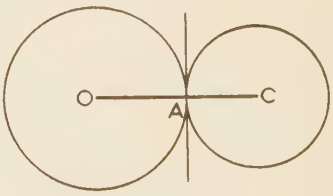
$\therefore$ OA is $\perp$ to RT (87).

204. Corollary. The perpendicular to a tangent at the point of contact passes through the center of the circle. (43.)

Q.E.D.

Proposition X. Theorem

205. If two circles are tangent to each other, the line joining their centers passes through their point of contact.



Given: ⓢ O and C tangent to a line at A, and line OC.

To Prove: OC passes through A.

Proof: Draw radii OA and CA.

OA is $\perp$ to the tangent and CA is $\perp$ to the tangent (203).

$\therefore$ OAC is a straight line (43).

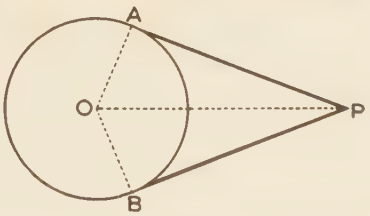
$\therefore$ OAC and OC coincide, and OC passes through A (39).

Q.E.D.

Let the pupil supply the proof if the circles are tangent *internally*.

Proposition XI. Theorem

206. Two tangents drawn to a circle from an external point are equal.



Note. In this theorem the word "tangent" signifies the distance between the external point and the point of contact.

Given: $\odot O$ and tangents PA, PB.

To Prove: Distance PA = distance PB.

Proof: Draw radii to points of contact, and join OP.

$\angle OAP$ and OBP are right $\measuredangle$ (203).

In rt. $\triangle OAP$ and OBP, $OP = OP$ (?); $OA = OB$ (?).

$\therefore \triangle OAP$ is congruent to $\triangle OBP$ (84).

$\therefore PA = PB$ (?).

Q.E.D.

Historical Note. Pythagoras, a Greek philosopher, born probably at Samos, in the sixth century B.C., had the reputation of being an "assiduous inquirer," and of having a great fund of general knowledge. He was a moral reformer as well as a scientific teacher. He was the head of a secret society the members of which were pledged to the severest discipline, and to the practice of temperance, purity, and obedience. The study of mathematics in Greece was magnified by him and advanced to the rank of a science. The invincible proof given on page 204 of the theorem that the square on the hypotenuse of a right triangle is equal in area to the sum of the squares on the legs, has been attributed to Pythagoras, and is often referred to as the Pythagorean proposition. The discovery of incommensurable magnitudes (p. 96) and of many other theorems and problems is ascribed to him.

Pythagoras

Proposition XII. Theorem

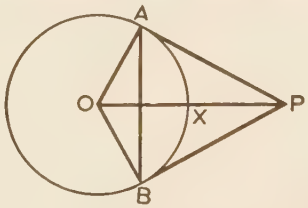
207. If from an external point tangents are drawn to a circle, and radii are drawn to the points of contact, the line joining the center and the external point bisects:

I. The angle formed by the tangents.

II. The angle formed by the radii.

III. The chord joining the points of contact.

IV. The arc intercepted by the tangents.



Given: Tangents AP and BP from point P and radii OA and OB.

To Prove: Line OP bisects:

I. $\angle APB$, II. $\angle AOB$, III. Chord AB, IV. Arc AXB.

Proof: $\triangle OAP$ and OBP are rt. $\triangle$ (?).

They are congruent. (Explain.)

I.	$\therefore \angle APO = \angle BPO$	(?).
II.	$\angle AOP = \angle BOP$	(?).
III.	O is equidistant from A and B	(?).
	P is also equidistant from A and B	(206).
	$\therefore OP$ is $\perp$ to AB at its midpoint	(83).
IV.	Arc AX = arc BX	(193).

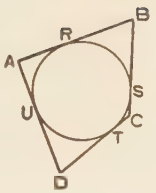
Q.E.D.

Ex. 1. Tangents drawn to a circle at the extremities of a diameter are parallel.

Ex. 2. Tangents drawn to a circle at the extremities of a chord form, with the chord, an isosceles triangle.

Ex. 3. The bisector of the angle between two tangents to a circle passes through the center.

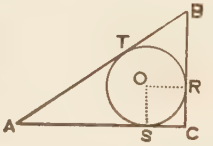
Ex. 4. The sum of one pair of opposite sides of a circumscribed quadrilateral is equal to the sum of the other pair.



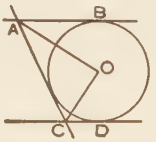
Ex. 5. A circumscribed parallelogram is equilateral.

Ex. 6. A circumscribed rectangle is a square.

Ex. 7. If a circle is inscribed in a right triangle, the sum of the diameter and the hypotenuse is equal to the sum of the legs.

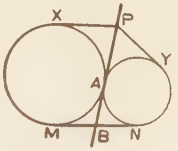


Ex. 8. If two parallel tangents meet a third tangent, and lines are drawn from the points of intersection to the center, they are perpendicular.



Ex. 9. Tangents drawn to two tangent circles from any point in their common interior tangent are equal.

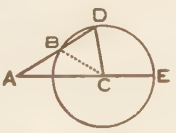
Ex. 10. The common interior tangent of two tangent circles bisects their common exterior tangent.



Ex. 11. Do the theorems of Ex. 9 and 10 apply if the circles are tangent internally? If so, prove.

Ex. 12. In the adjoining figure, if AE and AD are secants, AE passing through the center, and the external part of AD being equal to a radius, the angle $DCE = 3 \angle A$.

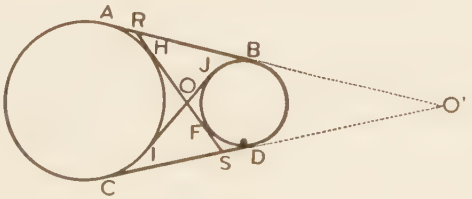
[Draw BC. $\angle DBC$ = ext. $\angle$ of $\triangle ABC = 2 \angle A = \angle D$. (Explain.) $\angle DCE$ = an ext. $\angle$, etc.]



Ex. 13. The two common interior tangents of two circles are equal.

Ex. 14. The common exterior tangents to two circles are equal.

[Produce them to intersection.]



Ex. 15. In the preceding figure, prove that $RH = SF$.

Proof: $AR + RB = CS + SD$;

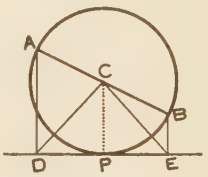
$\therefore AR + (RH + HF) = (SF + HF) + SD.$

$\therefore RH + RH + HF = SF + HF + SF$; $\therefore 2\,RH = 2\,SF$, etc. Give reasons and explain.

Ex. 16. The common exterior tangents to two circles intercept on a common interior tangent (produced), a line equal to a common exterior tangent. **To Prove:** $RS = AB$.

Ex. 17. AB and AC are two tangents from A; in the less arc BC a point D is taken and a tangent drawn at D, meeting AB at E and AC at F. Prove that $AE + EF + AF$ remains a uniform length for all positions of D in arc BC.

Ex. 18. If perpendiculars are drawn upon a tangent from the ends of any diameter:



(1) The point of tangency bisects the line between the feet of the perpendiculars.

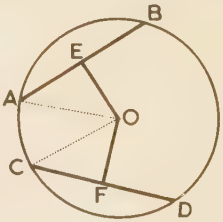
[Draw CP.]

(2) The sum of the perpendiculars equals the diameter.

(3) The center of the circle is equally distant from the feet of the perpendiculars.

Proposition XIII. Theorem

208. In the same circle (or in equal circles) equal chords are equally distant from the center.



Given: $\odot O$; chord AB = chord CD, and distances OE and OF.

To Prove: $OE = OF$.

Proof: Draw radii OA and OC.

In the rt. $\triangle$ AOE and COF,

	$AE = \frac{1}{2}\,AB$ and $CF = \frac{1}{2}\,CD$	(200).
But	$AB = CD$	(Hyp.).
	$\therefore AE = CF$	(Ax. 3).
Also	$AO = CO$	(?).
	$\therefore \triangle AOE$ is congruent to $\triangle COF$	(84).
	$\therefore OE = OF$	(?).

Q.E.D.

Proposition XIV. Theorem

209. In the same circle (or in equal circles) chords which are equally distant from the center are equal. [Converse.]

Given: $\odot O$; chords AB and CD; distance OE = distance OF.

To Prove: Chord AB = chord CD.

Proof: Draw radii OA and OC.

In rt. $\triangle AOE$ and COF,	$AO = CO$	(?).
Also	$EO = OF$	(Hyp.).
	$\therefore \triangle AOE$ is congruent to $\triangle COF$	(84).
	$\therefore AE = CF$	(?).
Now	AB is twice AE and CD is twice CF	(200).
	$\therefore AB = CD$	(Ax. 3). Q.E.D.

Ex. 1. If two circles are concentric, all chords of the greater that are tangent to the less are equal.

Ex. 2. If at the midpoint of an arc a tangent is drawn, it is parallel to the chord of the arc.

Ex. 3. If two equal chords intersect on the circle, the radius drawn to their point of intersection bisects their angle.

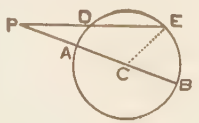
Ex. 4. If the line joining the point of intersection of two chords and the center bisects the angle formed by the chords, they are equal.

Ex. 5. The radius of the circle inscribed in an equilateral triangle is half the radius of the circle circumscribed about it. [Use 143.]

Ex. 6. If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.

Ex. 7. If two circles are concentric and a secant cuts them both, the portions of the secant intercepted between the circumferences are equal.

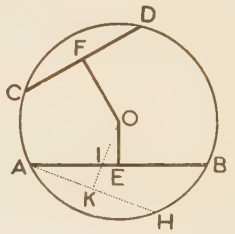
Ex. 8. Of all secants that can be drawn to a circumference from a fixed external point, the longest passes through the center.



Ex. 9. The shortest line from an external point to a circumference is that which, if produced, would pass through the center.

Proposition XV. Theorem

210. In the same circle (or in equal circles) if two chords are unequal, the greater chord is at the less distance from the center.



Given: $\odot O$; chord $AB >$ chord CD, and distances OE and OF.

To Prove: $OE < OF$.

Proof: Arc $AB >$ arc CD (198, I).

Suppose arc AH is taken on arc AB, equal to arc CD. Draw chord AH. Draw $OK \perp$ to AH, cutting AB at I.

Now	chord $AH =$ chord CD	(197).
	$\therefore$ distance $OK =$ distance OF	(208).
But	$OE < OI$	(87).
Also	$OI < OK$	(Ax. 5).
	$\therefore OE < OK$	(Ax. 11).
Substituting,	$OE < OF$	(Ax. 6).

Q.E.D.

Proposition XVI. Theorem

211. In the same circle (or in equal circles) if two chords are unequally distant from the center, the chord at the less distance is the greater. [Converse.]

Given: $\odot O$; chords AB and CD; distance $OE <$ distance OF.

To Prove: Chord $AB >$ chord CD.

Proof: It is evident that chord $AB <$ CD, or $=$ chord CD, or $>$ chord CD. Proceed by the method of exclusion.

Another Proof: On OF take $OX =$ to OE. At X draw a chord $RS \perp$ to OX.

Then	chord RS is $\parallel$ to chord CD.	(62).
	$\therefore$ arc $RS >$ arc CD	(Ax. 5).

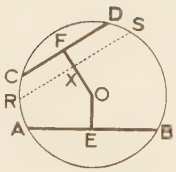
$\therefore$ chord $RS >$ chord CD (198, II).

But chord $AB =$ chord RS (209).

Substituting,

chord $AB >$ chord CD (Ax. 6).

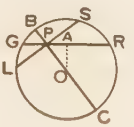
Q.E.D.



212. Corollary. The diameter of a circle is longer than any other chord.

Ex. 1. What is the longest chord that can be drawn through a given point within a circle?

Ex. 2. Of all chords that can be drawn through a given point within a circle, the chord perpendicular to the diameter through the given point is the shortest.



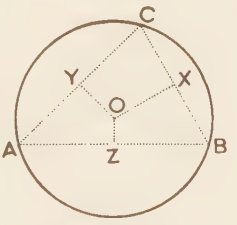
Given: P, the point; BOC the diam.; $LS \perp$ to BC at P; GR, any other chord through P.

To Prove: (?)

Proof: Draw $OA \perp$ to GR. Etc.

Proposition XVII. Theorem

213. Through three points, not in the same straight line, one circle can be drawn, and only one.



Given: Points A and B and C.

To Prove: I. (?). II. (?).

Proof: I. Draw lines AB, BC, AC. Suppose their $\perp$ bisectors, OZ, OX, OY, are drawn. These $\perp$s will meet at a point (100).

With O as a center and OA or OB or OC as a radius, a circle can be described through A, B, and C (100).

II. These $\perp$s meet at only one point (100).

That is, there is only one center.

The distances OA, OB, OC are all equal (100).

That is, there is only one radius.

$\therefore$ there can be only one circle (192). Q.E.D.

214. Corollary. One circle, and only one, can be drawn through the vertices of a triangle.

215. Corollary. A circle is determined by three points.

216. Corollary. A circle cannot be drawn through three points which are in the same straight line.

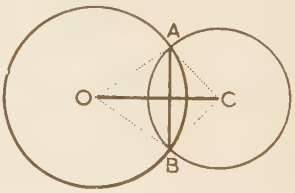
[The ⫡ would be ∥.]

217. Corollary. A straight line can intersect a circle in only two points. (216.)

218. Corollary. Two circles can intersect in only two points.

Proposition XVIII. Theorem

219. If two circles intersect, the line joining their centers is the perpendicular bisector of their common chord.



Given: (?).

To Prove: (?).

Proof: Draw radii in each ⊙ to ends of AB.

Point O is equally distant from A and B (187).

Point C is equally distant from A and B (?).

$\therefore$ OC is the ⊥ bisector of AB (83).

Q.E.D.

Ex. 1. Illustrate the five corollaries on this page by diagrams.

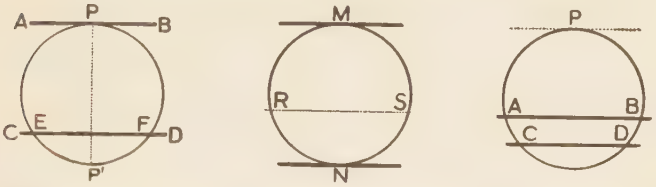
Ex. 2. On an island six miles from the mainland is a gun having a range of ten miles. Draw a diagram, using a scale of $\frac{1}{8}$ in. to the mile, showing the range of the gun.

Ex. 3. On the opposite sides of the entrance to a harbor are two forts, twelve miles apart. In each there is a gun with a range of nine miles. Draw a diagram showing the region exposed to the fire of each gun, and to the fire of both guns.

Ex. 4. Make a similar problem using three forts, and guns of different ranges, and draw the diagram, showing regions exposed to one gun only and to all three guns (if any).

Proposition XIX. Theorem

220. Parallel lines intercept equal arcs on a circle.



Given: A circle and a pair of parallels intercepting two arcs.

To Prove: The intercepted arcs are equal.

There may be three cases:

I. If the ∥s are a tangent (AB, tangent at P) and a secant (CD, cutting the circle at E and F).

Proof: Draw diameter to point of contact, P.

This diameter is $\perp$ to AB (203).

PP' is also $\perp$ to EF (64).

$\therefore$ arc EP = arc FP (199).

II. If the ∥s are two tangents, points of contact being M and N.

Proof: Suppose a secant is drawn ∥ to one of the tangents, cutting the ⊙ at R and S.

RS will be ∥ to the other tangent (63).

$\therefore$ arc MR = arc MS and arc RN = arc SN (I).

Adding, arc MRN = arc MSN (Ax. 2).

III. If the ∥s are two secants, one cutting the ⊙ at A and B, the other at C and D.

Proof: Suppose a tangent is drawn, ∥ to AB and touching the ⊙ at P. This tangent is ∥ to CD (63).

Arc PC = arc PD and arc PA = arc PB (I).

Subtracting, arc AC = arc BD (Ax. 2).

Q.E.D.

221. A **polygon** is **inscribed** in a circle, or a **circle** is **circumscribed** about a polygon if the vertices of the polygon are in the circle, and its sides are chords.

A **polygon** is **circumscribed** about a circle, or a **circle** is **inscribed** in a polygon if the sides of the polygon are all tangent to the circle.

The **perimeter** of a figure is the sum of all its bounding lines.

Exercises in Drawing Circles

1. Draw two unequal intersecting circles. Show that the line joining their centers is less than the sum of their radii.

2. Draw two circles externally (not tangent) and show that the line joining their centers is greater than the sum of their radii.

3. Draw two circles tangent externally. Discuss these lines similarly.

4. Draw two circles tangent internally. Discuss these lines similarly.

5. Draw two circles so that they can have only one common tangent.

6. Draw two circles so that they can have two common tangents.

7. Draw two circles so that they can have three common tangents.

8. Draw two circles so that they can have four common tangents.

9. Draw two circles so that they can have no common tangent.

SUMMARY

222. The following is a summary of the truths relating to magnitudes, which have been already established in Book II.

I. **Arcs are equal** if they are:

(1) Intercepted by equal central angles.
(2) Subtended by equal chords.
(3) Intercepted by parallel lines.
(4) Halves of the same arc, or of equal arcs.

II. **Lines are equal** if they are:

(1) Radii of the same or of equal circles.
(2) Diameters of the same or of equal circles.
(3) Chords that subtend equal arcs.
(4) Chords that are equally distant from the center.
(5) Tangents to one circle from the same point.

III. **Unequal arcs** and **unequal chords** have like relations.

ORIGINAL EXERCISES

1. Show that an inscribed trapezoid is isosceles.

2. In the figure of 242, show that arc DBX = arc DB + arc AC.

3. In the figure of 243, show that arc CX = arc CMB − arc CNB.

4. In the figure of 244, show that arc CX = arc CE − arc BD.

5. In the figure of 245, show that arc BX = arc BE − arc BD.

6. Show that the perpendiculars to the sides of a circumscribed polygon at the points of contact meet at a common point.

7. Show that the bisectors of the angles of a circumscribed polygon meet at a common point.

8. If two circles intersect and the four radii are drawn to the points of intersection, prove that the line joining the centers of the circles bisects the central angles formed by these radii.

9. If two chords of a circle are equal, but not parallel, and their midpoints are joined by a line, prove that the line from the center of the circle to the midpoint of the other line is perpendicular to it.

10. If a hexagon is circumscribed about a circle, prove that the sum of three alternate sides equals the sum of the other three sides.

11. Draw two circles which can have neither a common chord nor a common tangent.

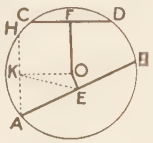
12. Two perpendicular radii are prolonged to meet a tangent to a circle, and from the two points of intersection two other tangents are drawn to this circle. Prove that these two tangents are parallel.

(Hint. Draw radii to the three points of contact.)

13. If from the midpoint of an arc perpendiculars are drawn to the radii drawn to the ends of the arc, prove that these perpendiculars are equal.

14. If through the extremities of a diameter two equal chords are drawn, one on each side of the diameter, prove that they are parallel.

15. Prove the theorem of 210 by the accompanying figure.



[Hint. in $\triangle AEK$ show that two sides are unequal, hence two $\measuredangle$ are unequal, hence two angles in $\triangle EKO$ are unequal, etc.]

16. Prove the theorem of 211 by this figure, and a method similar to that employed in Ex. 15.

KINDS OF QUANTITIES — MEASUREMENT

223. A **ratio** is the quotient of one quantity divided by another — both being of the same kind.

224. To **measure a quantity** is to find the number of times it contains another quantity of the same kind, called the **unit.** This *number* is the ratio of the quantity to the unit.

225. Two quantities are called **commensurable** if there exists a common unit of measure which is contained in each a whole (integral) number of times.

Two quantities are called **incommensurable** if there does not exist a common unit of measure which is contained in each a whole number of times.

Thus, \$17 and \$35 are commensurable, but \$17 and $\$\sqrt{35}$ are incommensurable. Two lines $18\frac{1}{2}$ ft. and 13 yd. are commensurable, but $18\frac{1}{2}$ ft. and $\sqrt[3]{13}$ yd. are incommensurable.

226. A **constant** quantity is a quantity the value of which does not change during a discussion. A constant may have only one value.

A **variable** is a quantity that has different successive values during a discussion. It may have an unlimited number of values.

227. The **limit of a variable** is a constant, *to which* the variable cannot be equal, but *from which* the variable can be made to differ by less than any mentionable quantity.

228. **Illustrative.** The *ratio* of 15 yd. to 25 yd. is written either $\frac{15}{25}$ or $15 \div 25$ and is equal to three fifths. If we state that a son is two thirds as old as his father, we mean that the son's age divided by the father's, equals two thirds. A ratio is a fraction.

The statement that a certain distance is 400 yd. signifies that the unit (the yard), if applied to this distance, will be contained exactly 400 times.

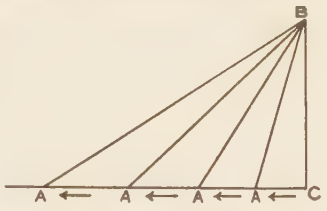
Are \$7.50 and \$3.58 *commensurable* if the unit is \$1? 1 dime? 1 cent?

Are 10 ft. and $\sqrt{19}$ ft. *commensurable?*

The height of a steeple is a *constant;* the length of its shadow made by the sun is a *variable.* The distance a train goes *varies* with the time it travels. Our ages are *variables.* The length of a standard yard, mile,

or meter, etc., is a *constant*. The height of a growing plant or a child is a *variable*.

The *limit* of a variable may be illustrated by considering a right triangle ABC, and supposing the vertex A to move farther and farther from the vertex of the right angle. It is evident that the hypotenuse becomes longer, that AC increases, but BC remains the same length. The angle A decreases, the angle B increases, but the angle C remains constantly a right angle. If we carry vertex A toward the left indefinitely, the $\angle A$ becomes less and less but cannot become zero. [Because then there could be no $\triangle$.]



Hence the limit of the decreasing $\angle A$ is zero.

Likewise, the $\angle B$ becomes larger and larger but cannot become equal to a right angle. [Because then two sides of the triangle would be parallel, which is impossible.] But it may be made as nearly equal to a right angle as we choose.

Hence the limit of $\angle B$ is a right angle.

To these limits we cannot make the variables equal, but *from* these limits we can make them differ by less than any mentionable angle, however small.

The following supplies another illustration of the limit of a variable. The sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} +$ etc., will always be less than 2, no matter how many terms are collected. But by taking more and more terms we can make the actual difference between this sum and 2 less than any conceivable fraction, however small. Hence 2 is the limit of the sum of the series. The limit is not 3 or 4, because the difference between the sum and 3 cannot be made less than any assigned fraction. Neither is the limit $1\frac{1}{3}$. (Why not?) Similarly, the limit of the value of .333333 ····· *ad infinitum* is $\frac{1}{3}$.

Certain variables become equal to a fixed magnitude; but this fixed magnitude is not a limit. Thus, the length of the shadow of a tower really becomes equal to a fixed distance (at noon). A man's age really attains to a definite number of years and then ceases to vary (at death).

Hence if a variable approaches a constant, and the difference between the two can be made indefinitely small while the variable cannot become equal to the constant, the constant is the **limit of the variable**. This is merely another definition of a limit.

Proposition XX. Theorem of Limits

229. If two variables are always equal and each approaches a limit, their limits are equal.

Given: Two variables v and v'; v always $= v'$; also v approaching the limit l; v' approaching the limit l'.

To Prove: $l = l'$.

Proof: v is always $=$ to v' (Hyp.). Hence they may be considered as a single variable. Now a single variable can approach only one limit (228). Hence $l = l'$. Q.E.D.

230. (1) **Algebraic principles concerning variables.**

If v is a variable and k is a constant:

I. $v + k$ is a variable.

II. $v - k$ is a variable.

III. $k \pm v$ is a variable.

IV. kv is a variable.

V. $\frac{v}{k}$ is a variable.

VI. $\frac{k}{v}$ is a variable.

These six statements are obvious.

(2) **Algebraic principles concerning limits.**

If v is a variable whose limit is l, and k is a constant:

I. $v \pm k$ will approach $l \pm k$ as a limit.

II. $k \pm v$ will approach $k \pm l$ as a limit.

III. kv will approach kl as a limit.

IV. $\frac{v}{k}$ will approach $\frac{l}{k}$ as a limit.

V. $\frac{k}{v}$ will approach $\frac{k}{l}$ as a limit.

Note. A variable, as applied to Plane Geometry, is not added to, subtracted from, multiplied by, or divided by another variable.

Proofs: I. v cannot $= l$ (227). $\therefore v \pm k$ cannot $= l \pm k$.
Also, $v - l$ approaches zero (227).
$\therefore (v \pm k) - (l \pm k)$ approaches zero. (Because it reduces to $v - l$.)
Hence $v \pm k$ approaches $l \pm k$ (227).

II. Demonstrated similarly.

III. If $kv = kl$, then $v = l$ (Ax. 3). But this is impossible (227).
$\therefore kv$ cannot $= kl$.

Also $v - l$ approaches zero (227).

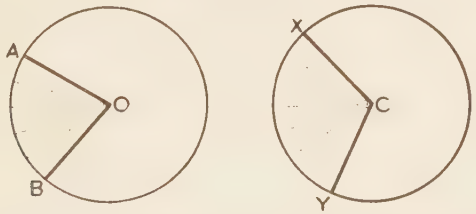
$\therefore k(v - l)$ or $kv - kl$ approaches zero.

Therefore kv approaches kl (227).

IV and V. Demonstrated similarly.

Proposition XXI. Theorem

231. In the same circle (or in equal circles) the ratio of two central angles is equal to the ratio of their intercepted arcs.



Given: $\odot O = \odot C$; central $\angle O$ and C; arcs AB and XY.

To Prove: $\dfrac{\angle O}{\angle C} = \dfrac{\text{arc } AB}{\text{arc } XY}$.

Proof: I. If the arcs are **commensurable**. There exists a common unit of measure of AB and XY (225).

Suppose this unit, when applied to the arcs, is contained 5 times in AB and 7 times in XY.

$$\therefore \frac{\text{arc } AB}{\text{arc } XY} = \frac{5}{7} \qquad \text{(Ax. 3).}$$

Draw radii to the several points of division of the arcs. $\angle O$ is divided into 5 parts and $\angle C$ into 7 parts.

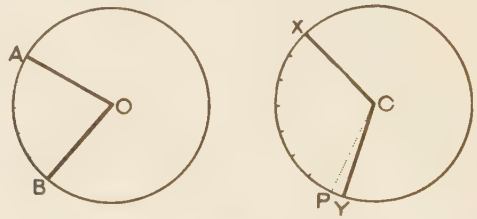
These 12 parts are all equal (194).

$$\therefore \frac{\angle O}{\angle C} = \frac{5}{7} \qquad \text{(Ax. 3).}$$

$$\therefore \frac{\angle O}{\angle C} = \frac{\text{arc } AB}{\text{arc } XY} \qquad \text{(Ax. 1).}$$

Q.E.D.

II. If the arcs are **incommensurable**. There does not exist a common unit (225). Suppose arc AB is divided into equal parts (any number of them). Apply one of these as a unit of measure to arc XY. There is a remainder PY. (Because AB and XY are incommensurable.)



Draw CP. Now $\frac{\angle O}{\angle XCP} = \frac{\text{arc } AB}{\text{arc } XP}$ (Case I).

Indefinitely increase the number of subdivisions of arc AB. Then each part, that is, our unit or divisor, is indefinitely decreased. Hence PY, the remainder, is indefinitely decreased. (Because the remainder $<$ the divisor.)

That is, arc PY approaches zero as a limit.

and $\angle PCY$ approaches zero as a limit.

$\therefore$ arc XP approaches arc XY as a limit (227).

and $\angle XCP$ approaches $\angle XCY$ as a limit (227).

$\therefore \frac{\angle O}{\angle XCP}$ approaches $\frac{\angle O}{\angle XCY}$ as a limit

and $\frac{\text{arc } AB}{\text{arc } XP}$ approaches $\frac{\text{arc } AB}{\text{arc } XY}$ as a limit.

$$\therefore \frac{\angle O}{\angle XCY} = \frac{\text{arc } AB}{\text{arc } XY} \quad (229).$$

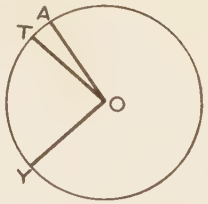
Q.E.D.

Ex. 1. If you double an arc do you double its central angle? its chord?

Ex. 2. If in two equal circles, an arc in one is taken three times as long as an arc in the other, how do their central angles compare? Is there any similar law that you know, applying to their chords?

Ex. 3. Two arcs of a circle contain 80° and 120° respectively. What is the ratio of their central angles?

Proposition XXII. Theorem



232. A central angle is measured by its intercepted arc.

Given: $\odot O$; $\angle AOY$; arc AY.

To Prove: $\angle AOY$ is measured by the arc AY, that is, they contain the same number of units.

Proof: The sum of all $\measuredangle$ about $O = 4$ rt. $\measuredangle = 360°$ (47).

If this $\odot$ is divided into 360 equal parts and radii are drawn to the several points of division, there will be 360 equal central $\measuredangle$ (194).

Each of these 360 central angles will be a *degree of angle* (20).

Each of the 360 equal arcs is called a *degree of arc.* Take $\angle AOT$, one of these degrees of angle, and arc AT, one of the degrees of arc. Then $\frac{\angle AOY}{\angle AOT} = \frac{\text{arc } AY}{\text{arc } AT}$ (231).

$\frac{\angle AOY}{\angle AOT} = \angle AOY \div$ a unit $=$ the number of units in $\angle AOY$ (224).

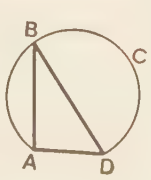
$\frac{\text{arc } AY}{\text{arc } AT} =$ arc $AY \div$ a unit $=$ the number of units in arc AY (224).

$\therefore$ the number of units in $\angle AOY =$ the number of units in arc AY (Ax. 1).

That is, $\angle AOY$ is measured by arc AY.

Q.E.D.

233. Corollary. **A central right angle intercepts a quadrant of arc.** (Because each contains 90 units.)

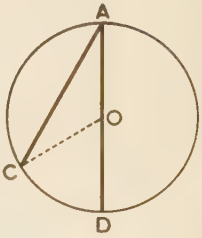


234. Corollary. **A right angle is measured by half a semicircle, that is, by a quadrant.**

235. An **angle** is **inscribed** in a segment if its vertex is on the arc and its sides are drawn to the ends of the arc of the segment.

Thus, $ABCD$ is a segment and $\angle ABD$ is inscribed in it.

PROPOSITION XXIII. THEOREM



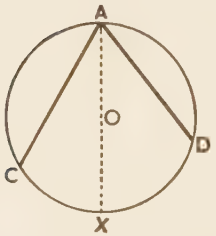
236. An inscribed angle is measured by half its intercepted arc.

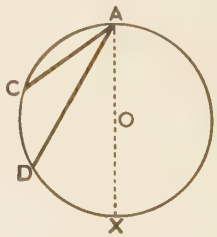
Given: $\odot O$; inscribed $\angle A$; arc CD.

To Prove: $\angle A$ is measured by $\frac{1}{2}$ arc CD.

Proof: I. If one side of the $\angle$ is a diameter.

Draw radius CO.	$\triangle AOC$ is isosceles	(?).
	$\angle COD = \angle A + \angle C$	(102).
	$\angle A = \angle C$	(?).
	$\therefore \angle COD = \angle A + \angle A = 2\angle A$	(Ax. 6).
That is,	$\frac{1}{2}\angle COD = \angle A$	(Ax. 3).
But	$\angle COD$ is meas. by arc CD	(232).
	$\therefore \frac{1}{2}\angle COD$ is meas. by $\frac{1}{2}$ arc CD	(Ax. 3).
	$\therefore \angle A$ is meas. by $\frac{1}{2}$ arc CD	(Ax. 6).





II. If the center is *within* the angle. Draw diameter AX.

$\angle CAX$ is measured by $\frac{1}{2}$ arc CX	(I).
$\angle DAX$ is measured by $\frac{1}{2}$ arc DX	(I). Adding,
$\therefore \angle CAD$ is measured by $\frac{1}{2}$ arc CD	(Ax. 2).

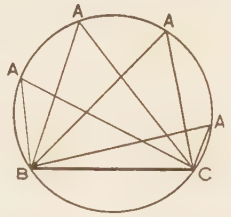
III. If the center is *without* the angle. Draw diameter AX.

$\angle CAX$ is measured by $\frac{1}{2}$ arc CX	(I).
$\angle DAX$ is measured by $\frac{1}{2}$ arc DX	(I). Subtracting,
$\therefore \angle CAD$ is measured by $\frac{1}{2}$ arc CD	(Ax. 2). Q.E.D.

237. Corollary. Angles measured by half the same arc, or halves of equal arcs, are equal.

238. Corollary. In the same circle (or in equal circles), equal angles are measured by equal arcs.

Proposition XXIV. Theorem



239. All angles inscribed in the same segment are equal.

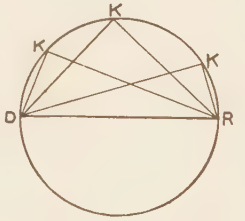
Given: The several $\measuredangle A$ inscribed in segment BAC.

To Prove: These angles all equal.

Proof: Each $\angle BAC$ is measured by $\frac{1}{2}$ arc BC (236).

$\therefore$ these angles are all equal. (237).

Q.E.D.



240. Corollary. All angles inscribed in a semicircle are right angles.

Proof: Each $\angle K$ is measured by $\frac{1}{2}$ a semicircle (236).

$\therefore$ each $\angle K =$ a rt. $\angle$ (234).

Q.E.D.

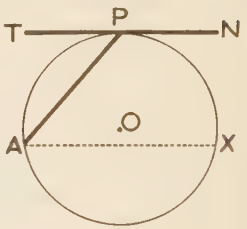
Historical Note. Thales, a Greek from Asia Minor, studied geometry from the Egyptians in the sixth century B.C. He discovered the truth of 240 as well as a number of very important theorems. For example: Book I, Propositions I, II, IV and XXXIV, and Book III, Proposition XX.

Thales was one of the "Seven Wise Men of Greece," and made important contributions to astronomy and philosophy as well as to geometry. He regarded water as the principle of all things.

Thales

Proposition XXV. Theorem

241. The angle formed by a tangent and a chord is measured by half the intercepted arc.



Given: $\odot O$, tangent TN; chord AP; $\angle TPA$; arc PA.

To Prove: $\angle TPA$ is measured by $\frac{1}{2}$ arc PA.

Proof: Through A suppose AX drawn $\parallel$ to TN.

Now $\angle A$ is meas. by $\frac{1}{2}$ arc PX		(236).
But	$\angle A = \angle TPA$	(66).
Also	arc PX = arc PA	(220).
Substituting, $\angle TPA$ is meas. by $\frac{1}{2}$ arc PA		(Ax. 6).

Q.E.D.

Ex. 1. A chord divides a circle into two arcs, one containing 100°, the other, 260°. An angle is inscribed in each segment. How many degrees are there in each angle?

Ex. 2. In a circle, an inscribed angle and a central angle intercept the same arc, which contains 140°. How many degrees are there in each angle?

Ex. 3. A chord subtends an arc of 74°. How many degrees are there in the angle between the chord and a tangent at one extremity?

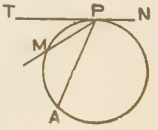
Ex. 4. The circumference of a circle is divided into four arcs, 40°, 70°, 100°, and x. Find x and the angles of the quadrilateral formed by the chords of these arcs.

Ex. 5. In a segment of a circle whose arc contains 210° is inscribed an angle. How many degrees are there in this angle?

Ex. 6. An inscribed angle contains 35°. How many degrees are there in its intercepted arc?

Ex. 7. The line bisecting an inscribed angle bisects also its intercepted arc.

Ex. 8. State and prove the converse of Ex. 7.

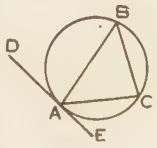


Ex. 9. The line bisecting the angle between a tangent and a chord bisects the intercepted arc.

Ex. 10. State and prove the converse of Ex. 9.

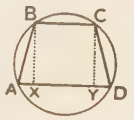
Ex. 11. The angle between a tangent and a chord is half the angle between the radii drawn to the ends of the chord.

Ex. 12. If a triangle is inscribed in a circle and a tangent is drawn at one of the vertices, the angles formed between the tangent and the sides equals the other two angles of the triangle.



Ex. 13. By the figure of Ex. 12 prove that the sum of the three angles of a triangle equals two right angles.

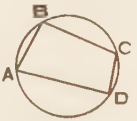
Ex. 14. If one pair of opposite sides of an inscribed quadrilateral are equal, the other pair are parallel.



Proof: Draw $\perp$s BX, CY; arc AB = arc CD (?).
$\therefore$ arc ABC = arc BCD (Ax. 2).
Hence prove rt. $\triangle ABX$ and DCY equal.

Ex. 15. If any pair of diameters is drawn, the lines joining their extremities (in order) form a rectangle.

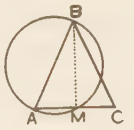
Ex. 16. The opposite angles of an inscribed quadrilateral are supplementary.



Ex. 17. If a tangent and a chord are parallel, and the chords of the two intercepted arcs are drawn, they make equal angles with the tangent.

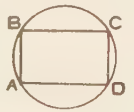
Ex. 18. The circle described on one of the equal sides of an isosceles triangle as a diameter bisects the base.

Proof: Draw line BM. The $\triangle$s are rt. $\triangle$ (?) and congruent (?).



Ex. 19. If the circle, described on a side of a triangle as diameter, bisects another side, the triangle is isosceles.

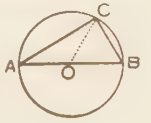
Ex. 20. All angles that are inscribed in a segment greater than a semicircle are acute, and all angles inscribed in a segment less than a semicircle are obtuse.



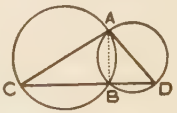
Ex. 21. An inscribed parallelogram is a rectangle.
Prove arc ABC = arc ADC, etc.

Ex. 22. The diagonal of an inscribed rectangle is a diameter.

Ex. 23. A circle described on the hypotenuse of a right triangle as a diameter passes through the vertices of all the right triangles having the same hypotenuse.



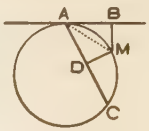
Ex. 24. If from one end of a diameter a chord is drawn, a perpendicular to it drawn from the other end of the diameter intersects the first chord on the circumference.



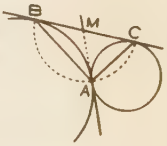
Ex. 25. If two circles intersect and a diameter is drawn in each circle through one of the points of intersection, the line joining the ends of these diameters passes through the other point of intersection. [Draw chord AB.]

Ex. 26. If a tangent is drawn at one end of a chord, the midpoint of the intercepted arc is equally distant from the chord and the tangent.

[Draw chord AM and prove the rt. $\triangle$ congruent.]

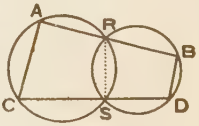


Ex. 27. If two circles are tangent at A and a common tangent touches them at B and C, the angle BAC is a right angle. [Draw tangent at A. Use 206, 240.]



Ex. 28. The bisectors of all the angles inscribed in the same segment of a circle pass through a common point.

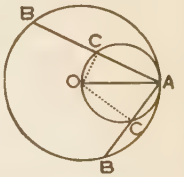
Ex. 29. If two circles intersect and a line is drawn through each point of intersection terminating in the circles, the chords joining these extremities are parallel. [Draw RS. $\angle A$ is supp. of $\angle RSC$ (?). Finally use 73.]



Ex. 30. A circle described on the radius of another circle as diameter bisects all chords of the larger circle drawn from their point of contact.

To Prove: AB is bisected at C.

Proof: Draw chord OC. (Use 240, 200).



Ex. 31. If two equal chords intersect within a circle, the segments of one are equal to the segments of the other, each to each.

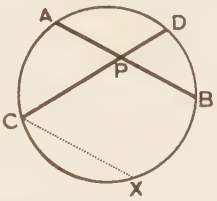
Ex. 32. Prove Proposition XXV by drawing a diameter to the point of tangency, instead of a chord parallel to the tangent.

Ex. 33. If in figure of Ex. 25 above, line CD met the two circles at M and N instead of at a single point B, what could be said of the lines AM and AN?

Ex. 34. If a circle is divided into four equal arcs and if chords of these arcs are drawn, the inscribed figure is a square.

Proposition XXVI. Theorem

242. The angle formed by two chords intersecting within the circle is measured by half the *sum* of the intercepted arcs. (The arcs are those intercepted by the given angle and by its vertical angle.)



Given: Chords AB and CD intersecting at P; $\angle APC$; arcs AC and DB.

To Prove: $\angle APC$ is measured by $\frac{1}{2}$ (arc AC + arc DB).

Proof: Suppose CX drawn through $C \parallel$ to AB.

Now $\angle C$ is measured by $\frac{1}{2}$ arc DX (236).

That is, $\angle C$ is measured by $\frac{1}{2}$ (arc BX + arc DB).

But $\angle C = \angle APC$ (66).

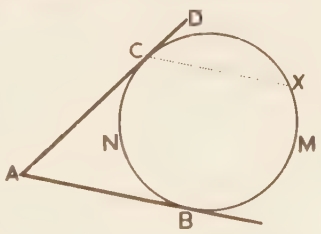
Also arc BX = arc AC (220).

$\therefore \angle APC$ is meas. by $\frac{1}{2}$ (arc AC + arc DB) (Ax. 6).

Q.E.D.

Proposition XXVII. Theorem

243. The angle formed by two tangents is measured by half the *difference* of the intercepted arcs.



Given: The two tangents AC and AB; $\angle A$; arcs CMB and CNB.

To Prove: $\angle A$ is measured by $\frac{1}{2}$ (arc CMB − arc CNB).

Proof: Suppose CX drawn $\parallel$ to AB.

Now $\angle DCX$ is meas. by $\frac{1}{2}$ arc CX (241).

That is, $\angle DCX$ is meas. by $\frac{1}{2}$ (arc CMB − arc BX).

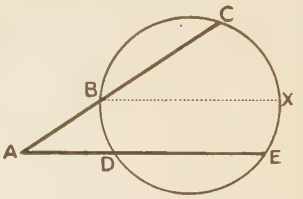
But $\angle DCX = \angle A$ (67).

Also arc BX = arc CNB (220).

$\therefore \angle A$ is meas. by $\frac{1}{2}$ (arc CMB − arc CNB) (Ax. 6).

Q.E.D.

Proposition XXVIII. Theorem



244. **The angle formed by two secants which intersect without the circle is measured by half the *difference* of the intercepted arcs.**

Given: (?).

To Prove: (?).

Proof: Suppose BX drawn. Where? How?

$\angle CBX$ is meas. by $\frac{1}{2}$ arc CX

That is, $\angle CBX$ is meas. by $\frac{1}{2}$ (arc CE − arc XE) (?).

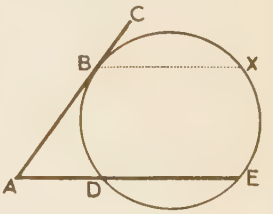
But $\angle CBX = \angle A$ (67).

And arc XE = arc BD (?).

Substituting, $\angle A$ is meas. by $\frac{1}{2}$ (arc CE − arc BD) (Ax. 6).

Q.E.D.

Proposition XXIX. Theorem



245. **The angle formed by a tangent and a secant which intersect without the circle is measured by half the *difference* of the intercepted arcs.**

Given: (?).

To Prove: (?).

Proof: Suppose BX drawn, etc.

$\angle CBX$ is meas. by $\frac{1}{2}$ arc BX (241).

That is, $\angle CBX$ is meas. by $\frac{1}{2}$ (arc BXE − arc XE).

But $\angle CBX = \angle A$ (?).

And arc XE = arc BD (?).

Substituting, $\angle A$ is meas. by $\frac{1}{2}$ (arc BXE − arc BD) (?).

Q.E.D.

Ex. 1. Where is the vertex of an angle that is measured by one arc? by half an arc? by half the sum of two arcs? by half the difference of two arcs?

Ex. 2. State these truths all in a single theorem of your own.

ORIGINAL EXERCISES

1. The arcs intercepted by two secants intersecting without a circle contain 20° and 140° respectively. How many degrees are there in the angle formed by the secants?

2. If in Ex. 1 the intersecting lines were chords, how many degrees would there be in their angle?

3. One of the arcs intercepted by two intersecting tangents is 72°. Find the angle formed by the tangents.

4. Two intersecting chords intercept opposite arcs of 28° and 80°. How many degrees are there in the angle formed by the chords?

5. The angle between a tangent and a chord contains 27°. How many degrees are there in the intercepted arc?

6. The angle between two chords is 30°; one of the arcs intercepted is 40°. Find the other arc. [Denote the arc by x.]

7. If in figure of 241, arc AP contains 124°, how many degrees are there in $\angle TPA$? in $\angle NPA$? in arc AX?

8. If in figure of 242, arc AC is 85°, $\angle APC$ is 47°, find arc DB.

9. If the arcs intercepted by two tangents contain 80° and 280°, find the angle formed by the tangents.

10. If the arcs intercepted by two secants contain 35° and 185°, find the angle formed by the secants.

11. If in figure of 243, arc CB is 155°, find the angle A.

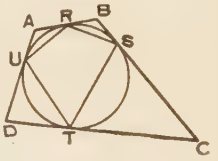
12. If in figure of 244, angle $A = 42°$ and arc $BD = 70°$, find arc CE.

13. If in figure of 245, angle $A = 18°$, arc $BXE = 190°$, find arc BD.

14. If the angle between two tangents is 80°, find the number of degrees in each intercepted arc. [Denote the arcs by x and $360° - x$.]

15. Three of the intercepted arcs of a circumscribed quadrilateral are 68°, 98°, 114°. Find the angles of the quadrilateral. If the chords are drawn connecting (in order) the four points of contact, find the angles of this inscribed quadrilateral. Also find the angles between the diagonals of this quadrilateral.

16. If the angle between two tangents to a circle is 40°, find the other angles of the triangle formed by drawing the chord joining the points of contact.



17. The circumference of a circle is divided into four arcs, three of which are, $RS = 62°$, $ST = 142°$, $TU = 98°$. Find:

(1) Arc UR.

(2) The three angles at R; at S; at T; at U.

(3) The angles A, B, C, D of circumscribed quadrilateral.

(4) The angles between the diagonals RT and SU.

(5) The angle between RU and ST at their point of intersection (if produced).

(6) The angle between RS and TU at their intersection.

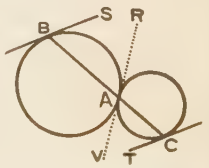
(7) The angle between AD and BC at their intersection.

(8) The angle between AB and DC at their intersection.

(9) The angle between RS and DC at their intersection.

(10) The angle between AD and ST at their intersection.

18. If in the figure of Ex. 17, $\angle A = 96°$; $\angle B = 112°$; and $\angle C = 68°$, find the angles of the quadrilateral $RSTU$. [Denote arc RU by x. $\therefore$ in $\triangle ARU$, $96° + \frac{1}{2}x + \frac{1}{2}x = 180°$. $\therefore x =$ etc.]

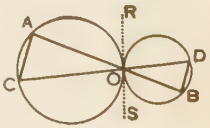


19. If two circles are tangent externally and any line through their point of contact intersects them at B and C, the tangents at B and C are parallel. [Draw common tangent at A. Prove: $\angle ACT = \angle ABS$.]

20. Prove the same theorem if the circles are tangent internally.

21. If two circles are tangent externally and any line is drawn through their point of contact terminating in the circles, the two diameters drawn to the extremities are parallel.

22. Prove the same theorem if the circles are tangent internally.



23. If two circles are tangent externally and any two lines are drawn through their point of contact intersecting the circles, the chords joining these points of intersection are parallel.

[Draw common tangent at O. Prove: $\angle C = \angle D$.]

24. Prove the same theorem if the circles are tangent internally.

25. Prove the theorem of 242 by drawing AD instead of CX, and using $\angle APC$ as an exterior angle of $\triangle APD$.

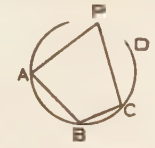
26. Prove the theorem of 243 by drawing BC and using $\angle DCB$ as an exterior angle of $\triangle ABC$.

27. Prove the theorem of 244 by drawing CD and using angle CDE as an exterior angle of triangle ACD.

28. Prove the same theorem by drawing BE.

29. Prove the theorem of 245 by drawing BE.

30. If the opposite angles of a quadrilateral are supplementary, a circle can be drawn circumscribing it.

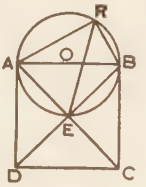


To Prove: A ⊙ can be drawn through A, B, C, P.

Proof: A ⊙ can be drawn through A, B, C (?). It is required to prove that it will contain point P. $\angle P + \angle B$ are supp. (?) $\therefore$ they must be meas. by half the entire circle. $\angle B$ is meas. by $\frac{1}{2}$ arc ADC (?). Hence $\angle P$ is meas. by $\frac{1}{2}$ arc ABC. If $\angle P$ is *within* or *without* the circle, it is not meas. by $\frac{1}{2}$ arc ABC. (Why not?)

31. The circle described on the side of a square, or of a rhombus, as a diameter passes through the point of intersection of the diagonals. [Use 135, 141.]

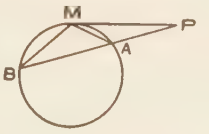
32. The line joining the vertex of the right angle of a right triangle to the point of intersection of the diagonals of the square constructed upon the hypotenuse as a side, bisects the right angle of the triangle.



Proof: Describe a ⊙ upon the hypotenuse as diameter and use 135, 196, 237.

33. If two secants, PAB and PCD, meet a circle at A, B, C, and D, respectively, the triangles PBC and PAD are mutually equiangular.

34. If PAB is a secant and PM is a tangent to a circle from P, the triangles PAM and PBM are mutually equiangular.



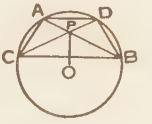
35. If two equal chords intersect within a circle,

(1) One pair of intercepted arcs are equal.

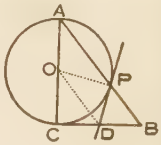
(2) Corresponding parts of the chords are equal.

(3) The lines joining their extremities (in order) form an isosceles trapezoid.

(4) The radius drawn to their intersection bisects their angle.

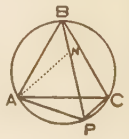


36. If a secant intersects a circle at D and E, PC is a parallel chord, and PR a tangent at P meeting the secant at R, the triangles PCD and PRD are mutually equiangular. [$\angle R$ and $\angle CDP$ are measured by $\frac{1}{2}$ arc PC. (Explain.) Etc.]

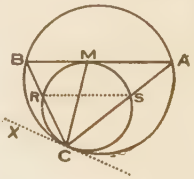


37. If a circle is described upon one leg of a right triangle as diameter and a tangent is drawn at the point of its intersection with the hypotenuse, this tangent bisects the other leg.

[Draw OP and OD. CD is tangent (?). OD bisects arc PC (207). $\angle COD = \angle A$ (237). $\therefore$ OD is $\parallel$ to AB (?). Etc.]



38. If an equilateral triangle ABC is inscribed in a circle and P is any point of arc AC, $AP + PC = BP$. [Take $PN = PA$; draw AN. $\triangle ANP$ is equilateral. (Explain.) $\triangle ANB = \triangle APC$ (?). Etc.]

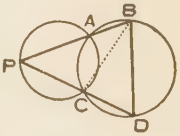


39. If two circles are tangent internally at C, and a chord AB of the larger circle is tangent to the less circle at M, the line CM bisects the angle ACB.

[Draw tangent CX and chord RS. (Explain.)

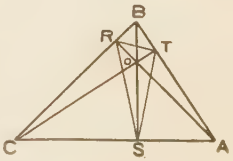
$$\angle RSC = \angle BCX = \angle A.$$

$\therefore$ AB is $\parallel$ to RS (?). Etc.]



40. If two circles intersect at A and C and lines are drawn from any point P, in one circle, through A and C terminating in the other at points B and D, chord BD will be of constant length for all positions of point P.

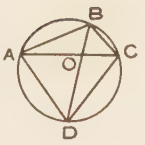
[Draw BC. Prove $\angle BCD$, the ext. $\angle$ of $\triangle PBC$, = a constant. Etc.]



41. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of these perpendiculars.

To Prove: BS bisects $\angle RST$, etc.

Proof: If a circle is described on AO as diam., it will pass through T and S (141). If a circle is described on OC as diam., it will pass through R and S (?). $\therefore$ $\angle BAR = BST$ (?); and $\angle BCT = \angle BSR$ (?). But $\angle BAR = \angle BCT$. (Each is the comp. of $\angle ABC$.) $\therefore$ Etc.



42. If ABC is a triangle inscribed in a circle, BD is the bisector of angle ABC, meeting AC at O and the circle at D, the triangles AOB and COD are mutually equiangular. Also triangles BOC and AOD. Also triangles BOC and ABD. Also triangles AOD and ABD. Also triangles BCD and COD.

43. If two circles intersect at A and B, and from P, any point on one of them, lines AP and BP are drawn cutting the other circle again at C and D respectively, CD is parallel to the tangent at P.

44. If two circles intersect at A and B, and through B a line is drawn meeting the circles at R and S respectively, the angle RAS is constant for all positions of the line RS.

[Prove $\angle R + \angle S$ is constant. $\therefore \angle RAS$ is also constant.]

45. Two circles intersect at A, and through A any secant is drawn meeting the circles again at M and N. Prove that the tangents at M and N meet at an angle which remains constant for all positions of the secant.

[Prove the angle between these tangents equal to the angle between the tangents to the circles at A.]

46. Two equal circles intersect at A and B, and through A any straight line MAN is drawn, meeting the circles at M and N respectively. Prove chord BM = chord BN.

47. If the midpoint of the arc subtended by any chord is joined to the extremities of any other chord,

(1) The triangles formed are mutually equiangular. (2) The opposite angles of the quadrilateral thus formed are supplementary.

48. Two circles meet at A and B and a tangent to each circle is drawn at A, meeting the circles at R and S respectively. Prove that the triangles ABR and ABS are mutually equiangular.

49. Two chords intersecting within a circle divide the circumference into parts that bear the relation to each other of 1, 2, 3, 4. Find the angles made by the chords. [Denote the arcs by x, $2x$, $3x$, $4x$.]

50. If $ABCD$ is an inscribed quadrilateral, AB and DC produced to meet at E, AD and BC produced to meet at F, the bisectors of angles E and F are perpendicular.

[The difference of one pair of arcs = difference of a second pair; the difference of a third pair = difference of a fourth pair. (Explain.) Transpose negative terms and add correctly, noting that the sum of 4 arcs = sum of 4 others, and hence = 180°. Half the sum of these 4 arcs measures the angle between the bisectors. (Explain.) Etc.]

LOCI

246. The **locus** of a point is the series of positions the point must occupy in order that it may satisfy a given condition. It is the path of a point whose positions are limited or defined by a given condition, or given conditions.

247. Explanatory. I. If a point is moving so that it is always one inch from a given point, the moving point may occupy any position in a circle whose center is the fixed point and whose radius is one inch. Furthermore, this moving point cannot occupy any position outside of the circle, or its position will not fulfill the given condition.

THEOREM. The locus of points in a plane a given distance from a given point is a circle the center of which is the given point and the radius of which is the given distance.

II. If a point is moving so that it is always equally distant from the ends of a straight line, it must move in the perpendicular bisector of the line.

THEOREM. **The locus of points equally distant from the ends of a line is the perpendicular bisector of the line.**

Proof: Every point *in* the ⊥ bisector is equally distant from the ends of the line. (80.)

No point *without* this ⊥ fulfills that condition. (81.)

∴ the ⊥ bisector is the locus. (246.)

III. THEOREM. The locus of points equally distant from the sides of an angle is the bisector of the angle.

Proof: Any point within the bisector of an angle is equally distant from the sides. (94.)

Any point equally distant from the sides of an angle lies in the bisector. (95.)

Hence all the points in the bisector fulfill the condition and there are no other points that fulfill it.

That is, the bisector is the locus, etc. Q.E.D.

IV. The method of proving that a certain line or a group of lines is the locus of points satisfying a given condition, consists in proving that *every point* in the line fulfills the given requirement, and that there is *no other point* that fulfills it. In the above illustrations it is evident that every point in the lines that were called the "locus," fulfilled the conditions of the case. It is evident also that there is no point outside these "loci" that does so fulfill the conditions. That is, these "loci" contain *all* the points described.

ORIGINAL EXERCISES ON LOCI

1. What is the locus of a point so moving that it is always two feet away from a given line?

2. What is the locus of a point so moving that it is always equally distant from two parallel lines?

3. What is the locus of points equally distant from two given fixed points?

4. If all the radii of a circle were drawn, what would be the locus of their midpoints?

5. If all possible lines were drawn from a vertex of a triangle and terminating in the opposite side, what would be the locus of their midpoints?

6. What is the locus of the midpoints of a series of parallel chords in a circle? Prove.

7. What is the locus of the midpoints of all chords of the same length in a given circle? Prove.

8. What is the locus of all points from which two equal tangents can be drawn to two circles which are tangent to each other?

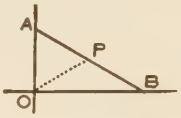
9. What is the locus of all points at a given distance from a given circumference? Discuss if the distance is $>$ radius. If it is less.

10. What is the locus of the vertices of the right angles of all the right triangles that can be constructed on a given hypotenuse? Prove.

11. What is the locus of the vertices of all the triangles which have a given acute angle (at that vertex) and have a given base? Prove.

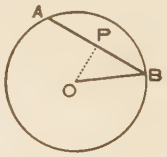
12. A line of given length moves so that its ends are in two perpendicular lines. What is the locus of its midpoint? Prove.

[Suppose AB represents one of the positions of the moving line. Draw OP to its midpoint. In all the positions of AB, $OP = \frac{1}{2}\ AB =$ a constant (141). $\therefore P$ is always a fixed distance from O. Etc.]



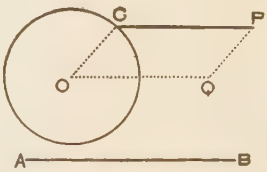
13. What is the locus of the midpoints of all the chords that can be drawn through a fixed point on a given circle?

[Suppose AB represents one of the chords from B in circle O, with radius OB; and P is the midpoint of AB. Draw OP. $\angle P$ is a rt. $\angle$ (?). That is, wherever the chord may be drawn, $\angle P$ is a rt. $\angle$. $\therefore$ locus of P is, etc.]



14. A definite line which is always parallel to a given line moves so that one of its extremities is on a given circle. Find the locus of the other extremity.

[Suppose CP represents one position of the moving line CP. Draw $OQ =$ and $\parallel$ to CP from center O. Join OC and PQ. Wherever CP is, this figure is a $\square$ (?). Its sides are of constant length (?). That is, P is always a fixed distance from Q, etc.]



15. What is the locus of the centers of all circles tangent to a given line at a given point? to a given circle at a given point?

16. A parallelogram, $ABCD$, is hinged at the vertices, and AB only is fixed in position. What is the locus of vertex C? of vertex D? of the midpoint of BC? of the midpoint of CD?

248. Heretofore only a few of the simplest exercises in construction have been given (pages 8–12), and formal proofs of these were not required.

The following methods for constructing lines are given so that mathematical precision may be employed in drawing accurate diagrams of a complex nature. No construction is considered valid unless a proof of its correctness can be given.

The pupil should be familiar with the use of the ruler and compasses.

CONSTRUCTION PROBLEMS

249. A geometrical **construction** is a diagram made of points and lines.

250. A geometrical **problem** is the statement of a required construction. Thus, "It is required to bisect a line" is a problem. A problem is sometimes defined as "a question to be solved" and includes other varieties besides those involved in geometry.

251. The word **proposition** is used to include both *theorem* and *problem.*

252. The complete solution of a problem consists of *five* parts:

I. The **Given** data are to be described.

II. The **Required** construction is to be stated.

III. The **Construction** is to be outlined.

This part usually contains the verb only in the *imperative.* The only limitation in this part of the process is that every construction demanded shall have been shown possible by previous constructions or postulates. (See 32, 33, 186.)

IV. The **Statement** that the required construction has been completed.

V. The **Proof** of this declaration.

Frequently a **discussion** of ambiguous or impossible instances is necessary.

253. NOTES. (1) A straight line is determined by two points.

(2) A circle is determined by three points.

(3) A circle is determined by its center and its radius. Whenever a circumference, or even an arc, is to be drawn, it is *essential* that the **center** and the **radius** be mentioned.

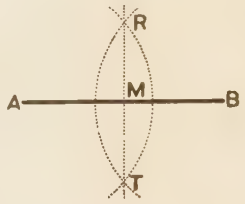
(4) "Q.E.F." = *Quod erat faciendum* — "which was to be done." These letters follow the statement that the construction which was required has been accomplished.

Proposition XXX. Problem

254. To bisect a given line.

Given: The definite line AB.

Required: To bisect AB.



Construction: Using A and B as centers and *one* radius, sufficiently long to make the circumferences intersect, describe two arcs meeting at R and T. Draw RT meeting AB at M.

Statement: Point M bisects AB. Q.E.F.

Proof:

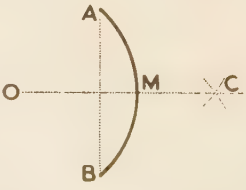
	R is equally distant from A and B	(188).
	T is equally distant from A and B	(?).
Hence	RT is the $\perp$ bisector of AB	(83).
That is,	M bisects AB.	Q.E.D.

Proposition XXXI. Problem

255. Problem. To bisect a given arc.

Given: Arc AB whose center is O.

Required: To bisect arc AB.



Construction: Draw chord AB. Using A and B as centers and any sufficient radius, describe arcs meeting at C. Draw OC cutting arc AB at M.

Statement: The point M bisects arc AB. Q.E.F.

Proof: O and C are each equally distant from A and B (188).

$\therefore$ OC is the $\perp$ bisector of chord AB (83).

$\therefore$ M bisects arc AB (200).

Q.E.D.

Ex. 1. Construct the supplement of a given angle.

Ex. 2. Divide a given line into four equal parts.

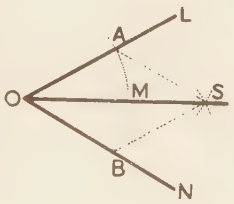
Ex. 3. Divide a given arc into four equal arcs.

Proposition XXXII. Problem

256. To bisect a given angle.

Given: $\angle LON$.

Required: To bisect $\angle LON$.



Construction: Using O as a center and any radius, draw arc AB, cutting LO at A and NO at B. Using A and B as centers and any sufficient radius, draw two arcs intersecting at S. Draw OS meeting arc AB at M.

Statement: OS bisects $\angle LON$. Q.E.F.

Proof: Draw AS and BS. In $\triangle AOS$ and BOS, $OS = OS$ (?).

$OA = OB$ and $AS = BS$ (188).

$\therefore \triangle AOS \cong \triangle BOS$ (?).

$\therefore \angle AOS = \angle BOS$ (?).

Q.E.D.

Proposition XXXIII. Problem

257. At a fixed point in a straight line to erect a perpendicular to that line.

Given: Line AB and point P within it.

Required: To erect a line $\perp$ to AB at P.

Construction: Using P as a center and any radius, draw arcs meeting AB at C and D. Using C and D as centers and a radius longer than before, draw arcs meeting at S. Draw PS.

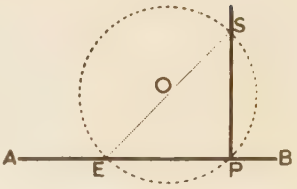
Statement: PS is $\perp$ to AB at P. Q.E.F.

Proof: Points P and S are each equally distant from C and D. (188).

$\therefore PS$ is the $\perp$ bisector of CD (83).

That is, PS is $\perp$ to AB. Q.E.D.

Another Construction: Using any point O, without AB, as center, and OP as radius, describe a circumference, cutting AB at P and E. Draw diameter EOS. Join SP.



Statement: SP is $\perp$ to AB at P. Q.E.F.

Proof: Segment SPE is a semicircle (191).

$\therefore \angle SPE$ is a rt. $\angle$ (240).

$\therefore SP$ is $\perp$ to AB (16). Q.E.D.

Ex. 1. Construct a right angle.

Ex. 2. Construct an angle of 45°; of 135°; of 22° 30′; of 67° 30′.

Ex. 3. Construct the complement of a given angle.

Ex. 4. Find the center of a given circle.

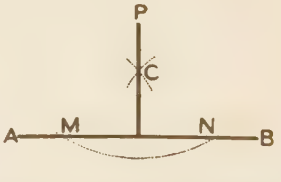
Ex. 5. Construct the second acute angle of a rt. $\triangle$ if one is known.

Ex. 6. Construct a chord of a circle if its midpoint is known.

Proposition XXXIV. Problem

258. Through a point without a line to draw a perpendicular to that line.

Given: Line AB and point P without it.



Required: (?).

Construction: Using P as a center and any sufficient radius, describe an arc intersecting AB at M and N. Using M and N as centers and any sufficient radius, describe arcs intersecting each other at C. Draw PC.

Statement: PC is $\perp$ to AB from P. Q.E.F.

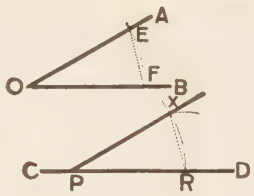
Proof: P and C are each equally distant from M and N (188).

$\therefore PC$ is the $\perp$ bisector of MN (83).

That is, PC is $\perp$ to AB from P. Q.E.D.

Proposition XXXV. Problem

259. At a given point in a given line to construct an angle which shall be equal to a given angle.



Given: $\angle AOB$; point P in line CD.

Required: To construct at P an $\angle$ = to $\angle AOB$.

Construction: Using O as a center with any radius, describe an arc cutting OA at E and OB at F. Draw chord EF. Using P as a center and OE as a radius, describe an arc cutting CD at R. Using R as a center and chord EF as a radius, describe an arc cutting the former arc at X. Draw PX and chord RX.

Statement: $\angle XPD = \angle AOB$. Q.E.F.

Proof: $OE = PX$ and $OF = PR$ and $EF = XR$ (188).

$\therefore \triangle OEF \cong \triangle PXR$ (?).

$\therefore \angle XPD = \angle O$ (?).

Q.E.D.

Proposition XXXVI. Problem

260. To draw a line through a given point parallel to a given line.

Given: Point P and line AB.

Required: To draw through P, a line $\parallel$ to AB.

Construction: Draw through P any line PN, meeting AB at N.

On this line, at P, construct $\angle NPX$ = to $\angle ANP$ (259).

Statement: PX is $\parallel$ to AB. Q.E.F.

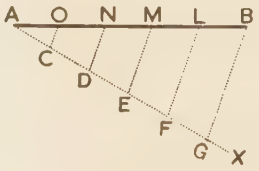
Proof: $\angle NPX = \angle ANP$ (Const.).

$\therefore PX$ is $\parallel$ to AB (70).

Q.E.D.

Proposition XXXVII. Problem

261. To divide a line into any number of equal parts.



Given: Definite line AB.

Required: To divide it into five equal parts.

Construction: Draw through A any other line AX. On this take any length AC as a unit, and mark off on AX five of these units, AC, CD, DE, EF, FG. Draw GB.

Through F, E, D, C, draw $\parallel$ to GB, lines FL, EM, DN, CO.

Statement: Then $AO = ON = NM = ML = LB$. Q.E.F.

Proof: $AC = CD = DE = EF = FG$ (Const.).

$\therefore AO = ON = NM = ML = LB$ (140).

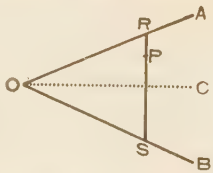
Q.E.D.

Ex. 1. Find a point in one side of a triangle equally distant from the other sides.

Ex. 2. Construct the shortest chord that can be drawn through a given point within a circle.

Proof: Draw any other chord through the point, etc.

Ex. 3. Draw through a given point without a given line, another line which shall make a given angle with the line.



Construction: Construct an $\angle$ at any point in the given line $=$ to given $\angle$, etc.

Ex. 4. Construct through a given point a line which makes equal angles with two intersecting lines.

Ex. 5. Bisect the three sides of a triangle, and draw the medians. Do they meet in a point?

Ex. 6. Draw accurately the three altitudes of a triangle. Do they meet in a point?

Ex. 7. Draw accurately the three perpendicular bisectors of the sides of a triangle. Do they meet in a point?

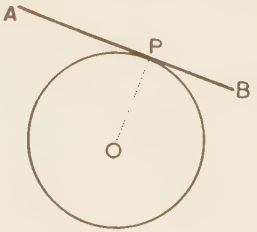
Ex. 8. Draw accurately the three bisectors of the angles of a triangle. Do they meet in a point?

Proposition XXXVIII. Problem

262. To draw a tangent to a given circle through a given point:

I. If the point is on the circle.

II. If the point is without the circle.



I. Given: $\odot O$; P, a point on the circle.

Required: To draw a tangent through P.

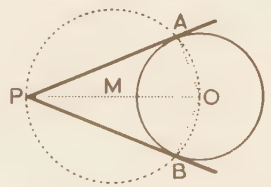
Construction: Draw the radius OP. Draw line $AB \perp$ to OP at P (by 257).

Statement: AB is tangent to $\odot O$ at P. Q.E.F.

Proof: AB is $\perp$ to OP at P (Const.).

$\therefore AB$ is a tangent (202). Q.E.D.

II. Given: $\odot O$; P, a point without it.



Required: To draw a tangent through P.

Construction: Draw PO; bisect it at M (by 254).

Using M as a center and PM as a radius, describe a circumference intersecting $\odot O$ at A and B.

Draw PA, PB, OA, OB.

Statement: PA and PB are tangents through P Q.E.F.

Proof: $\odot M$ passes through O ($PM = MO$ by const.).

$\therefore \angle PAO$ is a rt. $\angle$ (240).

$\therefore PA$ is a tangent (202).

Similarly, PB is a tangent. Q.E.D.

Note. The student has probably observed that in constructions certain lines and angles *must precede* others. The order of the successive steps is an important consideration. Thus, in 261 and 262 certain lines must be drawn before others *can* be drawn.

Proposition XXXIX. Problem

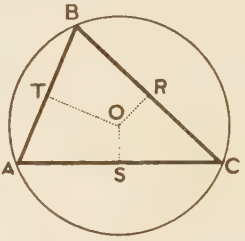
263. To circumscribe a circle about a given triangle.

Given: (?).

Required: (?). (See 214.)

Construction: Bisect AB, BC, AC. Erect $\perp$s at T, R, S, meeting at O.

Using O as a center and OA as radius, draw a circle.

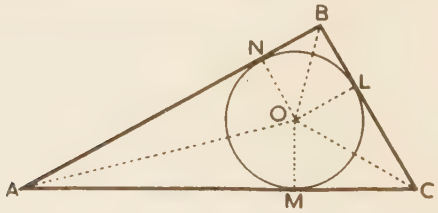


Statement: This $\odot$ will pass through vertices A, B, and C. Q.E.F.

Proof: (100). Q.E.D.

Proposition XL. Problem

264. To inscribe a circle in a given triangle.



Given: (?). **Required:** (?).

Construction: Draw the three bisectors of the $\measuredangle$s of $\triangle ABC$, meeting at O (by 256). Draw $\perp$ from O to the three sides.

Using O as a center and one $\perp$ as a radius, draw a $\odot$.

Statement: This $\odot$ will be tangent to the three sides of $\triangle ABC$. Q.E.F.

Proof: The bisectors of these angles meet in a point and the $\perp$s OL, OM, ON are equal (99).

$\therefore$ the circumference passes through L, M, N (179).

Also the three sides are tangent to the $\odot$ (202).

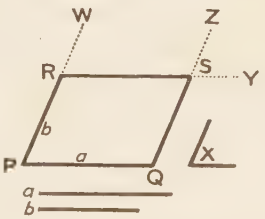
That is, the $\odot O$ is inscribed in $\triangle ABC$ (221).

Q.E.D.

265. If a circle is described tangent to one side of a triangle and tangent to the prolongations of the other sides, it is called an **escribed circle.** Every triangle may have three escribed circles.

Proposition XLI. Problem

266. To construct a parallelogram if two sides and the included angle are given.



Given: The sides a and b and their included angle, x.

Required: To construct a $\square$ containing these parts.

Construction: Take a straight line $PQ =$ to a.

At P construct $\angle P =$ to $\angle x$. On PW, the side of this $\angle$, take $PR =$ to b.

At R draw $RY \parallel$ to PQ; and at Q draw $QZ \parallel$ to PW.

Denote the intersection of these lines by S.

Statement: $PQSR$ is the required parallelogram. Q.E.F.

Proof: First, it is a parallelogram. (Def.).

Second, it is *the required* parallelogram. (Because it contains the given parts.) Q.E.D.

Ex. 1. Draw a triangle and all its exterior angles. Bisect these to find centers of escribed circles. What are the radii of these circles? Draw the three escribed circles.

Ex. 2. Draw a rectangle, having given two sides.

Ex. 3. What several things must be known about a circle before you can draw a tangent?

Ex. 4. Is a radius of a circle a tangent? Why?

Ex. 5. How can one find the center of a given arc? Is this the same as its midpoint?

Ex. 6. Give another method of solving Proposition XXXVII.

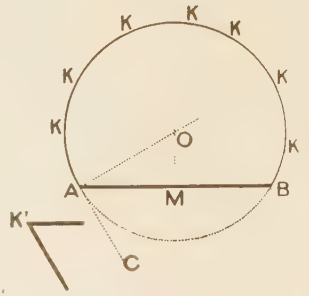
Ex. 7. If a hundred triangles stood on the same base, and all their vertex angles were equal, where would all their vertices be?

Proposition XLII. Problem

267. To construct a segment of a circle upon a given line, as chord, which shall contain angles equal to a given angle.

Given: Line AB and $\angle K'$.

Required: To construct a segment upon AB the inscribed angles of which shall $= \angle K'$.



Construction: Construct at A, $\angle BAC =$ to $\angle K'$.

Bisect AB at M.

At M erect $OM \perp$ to AB.

At A erect $OA \perp$ to AC, meeting OM at O.

Using O as a center and OA as radius, describe $\odot O$.

Statement: The $\measuredangle$ inscribed in segment $AKB = \angle K'$. Q.E.F.

Proof: The circle passes through B (80).

$\therefore AB$ is a chord (181).

AC is a tangent to the $\odot$ (202).

$\therefore \angle BAC$ is meas. by half the arc AB (241).

Any angle inscribed in AKB is meas. by half arc AB (236).

$\therefore$ any angle $AKB = \angle BAC$ (237).

$\therefore$ any inscribed $\angle AKB = \angle K'$ (Ax. 1).

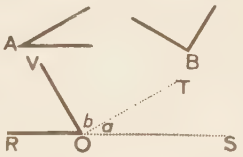
Q.E.D.

[Let the pupil draw chords AK and BK, which were purposely omitted.]

Historical Note. Substantial contributions were made to the advancement of geometrical science by Hippocrates, a Greek philosopher, who, during the fifth century B.C., discovered many properties of the circle. He was the first to employ the method of proof known as the *reductio ad absurdum*, and he wrote the first text book on geometry. He was not aware of the truth that equal central angles and equal inscribed angles intercept equal arcs, although he knew that areas of circles are proportional to the squares of their radii.

Proposition XLIII. Problem

268. To construct the third angle of a triangle if two angles are known.



Given: $\measuredangle A$ and B, two $\measuredangle$ of a $\triangle$.

Required: To construct the third.

Construction: At point O in line RS construct $\angle a =$ to $\angle A$.

At point O in OT construct $\angle b =$ to $\angle B$.

Statement: The $\angle VOR =$ the third $\angle$ of the $\triangle$. Q.E.F.

Proof: $\angle a + \angle b + \angle VOR = 2$ rt. $\measuredangle$ (46).

$\angle A + \angle B +$ the third $\angle$ of the $\triangle = 2$ rt. $\measuredangle$ (104).

$\therefore \angle a + \angle b + \angle VOR = \angle A + \angle B +$ the third $\angle$ (Ax. 1).

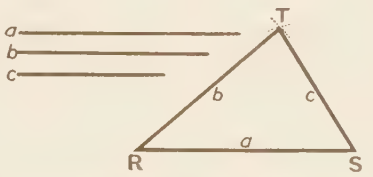
But $\angle a + \angle b = \angle A + \angle B$ (Const. and Ax. 2).

Subtracting, $\angle VOR =$ the third $\angle$ of the $\triangle$ (Ax. 2).

Q.E.D.

Proposition XLIV. Problem

269. To construct a triangle if the three sides are known.



Given: Sides a, b, c of a $\triangle$.

Required: To construct the $\triangle$.

Construction:

Draw $RS =$ to a.

Using R as a center and b as a radius, describe an arc. Using S as a center and c as a radius, describe an arc intersecting the former arc at T. Draw RT and ST.

Statement: $\triangle RST$ is the required $\triangle$. Q.E.F.

Proof: RST is a $\triangle$ (23).

RST is *the required* $\triangle$. (It contains a, b, c.) Q.E.D.

Discussion: Is this problem ever impossible? When?

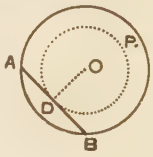
Ex. 1. Construct an angle of 60°; of 30°; of 15°; of 7° 30′; of 75°.

Ex. 2. Trisect a right angle.

Ex. 3. Construct a tangent to a circle, parallel to a given line.

Ex. 4. Construct a tangent to a given circle perpendicular to a given line.

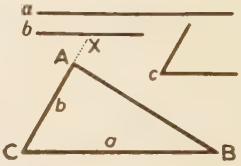
Ex. 5. Construct through a given point within a circle, two chords each equal to a given chord. Is this ever impossible?



Ex. 6. Construct in a given circle a chord equal to a second chord and parallel to a third.

Proposition XLV. Problem

270. To construct a triangle if two sides and the included angle are known.



Given: The sides a and b, and their included $\angle c$ in a $\triangle$.

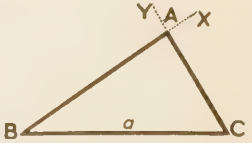
Required: To construct the $\triangle$.

Construction: Draw $CB =$ to a. At C construct $\angle BCX =$ to given $\angle c$. On CX take $CA =$ to b. Join AB.

Statement: (?). **Proof:** (?). **Discussion:** (?).

Proposition XLVI. Problem

271. To construct a triangle if a side and the two angles adjoining it are known.



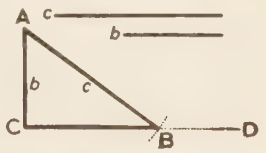
Given: (?). **Required:** (?).

Construction: Draw $BC =$ to a. At B construct $\angle CBX =$ to $\angle B'$; at C construct $\angle BCY =$ to $\angle C'$. Denote the point of intersection of BX and CY by A.

Statement: (?). **Proof:** (?). **Discussion:** (?).

Proposition XLVII. Problem

272. To construct a right triangle if the hypotenuse and a leg are known.



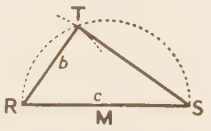
Given: Hypotenuse c; leg b.

Required: (?).

Construction: Draw an indefinite line CD and at C erect a $\perp =$ to b. Using A as a center and c as a radius, describe an arc cutting CD at B. Draw AB.

Statement: (?). **Proof:** (?). **Discussion:** (?).

Another Construction: Take a line, $RS =$ to c. With its midpoint M, as center, and RM as radius describe a semicircle. With R as center and b as radius, describe an arc cutting the semicircumference at T. Draw TR and TS.



Statement: (?). **Proof:** (?). **Discussion:** (?).

Proposition XLVIII. Problem

273. To construct a triangle if an angle, a side adjoining it, and the side opposite it are known; that is, if two sides and an angle opposite one of them are known.

The known angle may be obtuse, right, or acute. Consider:

First, If "side opposite" > "side adjoining."

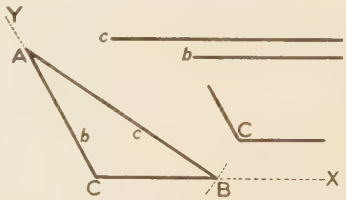
Second, If "side opposite" = "side adjoining."

Third, If "side opposite" < "side adjoining."

Construction for all: Draw an indefinite line, CX, and at one extremity construct an $\angle =$ to $\angle C$. Take on the side of this angle a distance from the vertex equal to the "side adjoining." Using the end of this side as a center and the "side opposite" as a radius, describe an arc intersecting CX. Draw radius to the intersection just found.

If the known angle is obtuse or right.

Given: $\angle C$, side c opposite it, and side b adjoining $\angle C$.



Construction: As above.

Discussion: *Case I.* $c > b$.

The $\triangle$ is always possible.

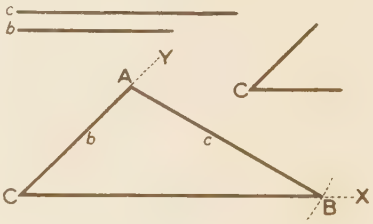
Case II. $c = b$.

The $\triangle$ is never possible (106).

Case III. $c < b$.

The $\triangle$ is never possible (116).

If the known angle is acute.



Case I. $c > b$.

The $\triangle$ is always possible.

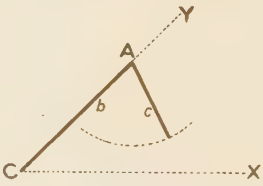
Case II. $c = b$.

The $\triangle$ is always possible and isosceles.

Case III. $c < b$.

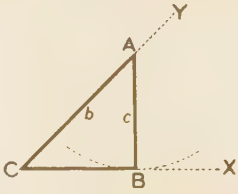
First, $c <$ the $\perp$ from A to CX.

The $\triangle$ is never possible.



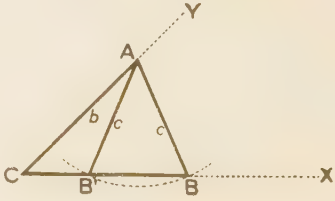
Second, $c =$ the $\perp$ from A to CX.

The $\triangle$ is possible and a right $\triangle$.



Third, $c >$ the $\perp$ from A to CX.

There are *two* $\triangle$, ACB and ACB'; both contain the three given parts.



ANALYSIS

Many constructions are so simple that their correct solution will readily occur to the pupil.

Sometimes, in the case of complicated constructions, the pupil must have the ability to put the given parts together, one by one. The following outline may be found helpful if employed intelligently. It is called the method of **analysis.**

I. Suppose the construction made, that is, suppose the figure drawn.

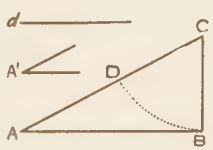
II. Study this figure in search of truths by which the **order** of the lines that have been drawn can be determined.

III. One or more auxiliary lines may be necessary.

IV. Finally, construct the figure and prove it correct.

EXERCISE. **Given the base of a triangle, an adjacent acute angle, and the difference of the other sides, to construct the triangle.**

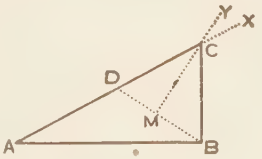
Given: Base AB; $\angle A'$; difference d.



Required: To construct the $\triangle$.

[**Analysis:** Suppose $\triangle ABC$ is the required $\triangle$. It is evident if $CD = CB$, they may be sides of an isos. $\triangle$ and $AD = d$.]

Construction: At A on AB construct $\angle BAX =$ to $\angle A'$ and on AX, take $AD =$ to d. Join DB. At M, midpoint of DB, draw $MY \perp$ to DB meeting AX at C. Draw CB.



Statement: (?). **Proof:** (?). **Discussion:** (?).

Historical Note. The philosopher Plato and his school flourished at Athens in the fourth century B.C. Plato brought to geometry exact definitions and axioms as well as the method of *analysis*, which is helpful in discovering difficult proofs and constructions.

ORIGINAL CONSTRUCTIONS

I. Construct an **isosceles triangle**, having given:

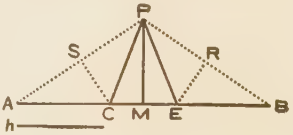
1. The base and one of the equal sides.
2. The base and one of the equal angles.
3. One of the equal sides and the vertex angle.
4. One of the equal sides and one of the equal angles.
5. The base and the altitude upon it.
6. The base and the radius of the inscribed circle.

[Bisect the base; erect a $\perp$ = to the radius; describe $\odot$, etc.]

7. The base and the radius of the circumscribed circle.
8. The altitude and the vertex angle.
9. The base and the vertex angle.

[Find the supplement of the given $\angle$; bisect this; at each end of base construct an $\angle$ = to this half; etc.]

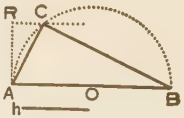
10. The perimeter and the altitude.



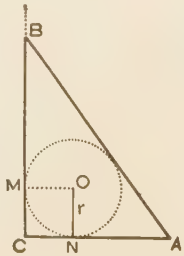
Given: Perimeter = AB; alt. = h. **Required**: (?). **Construction**: Bisect AB; erect at M $\perp$ = to h; draw AP and BP. Bisect these; erect $\perp$s SC and RE; etc.

II. Construct a **right triangle**, having given:

11. The two legs.
12. One leg and the adjoining acute angle.
13. One leg and the opposite acute angle.
14. The hypotenuse and an acute angle.
15. The hypotenuse and the altitude upon it.



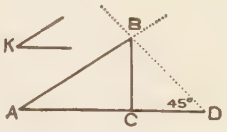
16. The median and the altitude upon the hypotenuse.
17. The radius of the circumscribed circle and a leg.
18. The radius of the inscribed circle and a leg.



Given: Radius = r; leg = CA. **Required**: (?). **Analysis**: Consider that ABC is the completed figure; $CNOM$ is a square, whose vertex O is the center of the circle, and side ON is the given radius. AB is tangent from A. **Construction**: On CA take CN = to r and construct square, $CNOM$. Prolong CM indefinitely. Describe $\odot$, etc.

19. One leg and the altitude upon the hypotenuse.

20. An acute angle and the sum of the legs.



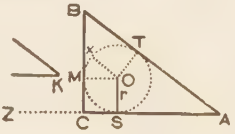
Given: AD = sum; $\angle K$. **Required**: (?). **Construction**: At A construct $\angle A$ = to $\angle K$; at D construct $\angle D$ = to 45°, etc.

21. The hypotenuse and the sum of the legs.

[Use A as center, hypotenuse as radius, etc.]

22. The radius of the circumscribed circle and an acute angle.

23. The radius of the inscribed circle and an acute angle.



Construction: Take CS = to r, on indefinite line, ZA. On CS construct square $CSOM$. At O construct $\angle MOX$ = to $\angle K$. Draw radius OT ⊥ to OX. Draw tangent at T, etc.

III. Construct an **equilateral triangle**, having given:

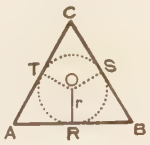
24. One side.

25. The altitude.

26. The perimeter.

27. A median.

28. The radius of the inscribed circle.



[Draw circle and radius; at center construct $\angle ROS$ = to 120° and $\angle ROT$ = to 120°; etc.]

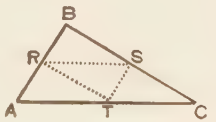
29. The radius of the circumscribed circle.

IV. Construct a **triangle**, having given:

30. The base, an angle adjoining it, the altitude upon it.

31. The midpoints of the three sides.

[Draw RS, RT, ST, etc.]



32. One side, altitude upon it, and the radius of the circumscribed circle.

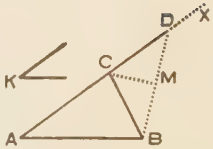
Construction: Draw ⊙ with given radius and any center. Take chord = to given side; etc.

33. One side, an adjoining angle, and the radius of the circumscribed circle.

34. Two sides and the altitude from the same vertex.

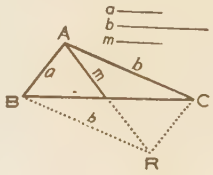
Construction: Erect ⊥ equal to altitude, upon an indefinite line. Use the end of this altitude as center, and the given sides as radii, etc.

35. One side, an angle adjoining it, and the sum of the other two sides.



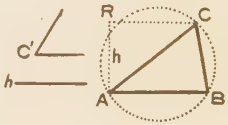
Construction: At A construct $\angle BAX =$ to given $\angle K$. On AX take $AD =$ to s; draw DB; bisect DB at M, etc.

36. Two sides and the median to the third side.



Given: a, b, m. **Construction**: Construct $\triangle ABR$ with three sides, $AB =$ to a, $BR =$ to b, $AR =$ to $2m$. Draw $AC \parallel$ to BR and $RC \parallel$ to AB meeting at C. Draw BC. **Statement**: (?). **Proof**: (?).

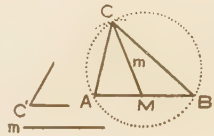
37. A side, the altitude upon it, and the angle opposite it.



Given: Side $=$ to AB, alt. $=$ to h; opposite $\angle C =$ to $\angle C'$.

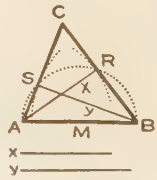
Construction: Upon AB construct segment ACB which contains $\angle C =$ to $\angle C'$ (by 267). At A erect $AR \perp$ to AB and $=$ to h; etc.

38. A side, the median to it, the angle opposite it.



[**Statement**: $\triangle ABC$ is the required $\triangle$.]

39. One side and the altitude from its extremities to the other sides.



Given: Side $= AB$, altitudes x and y.

Construction: Bisect AB; describe a semicircle. Using A as center and x as radius, describe arc cutting the semicircle at R; etc.

40. Two sides and the altitude upon one of them.

[**Given**: Sides $=$ to AB and BC; alt. on $BC =$ to x.]

41. One side, an angle adjoining it, and the radius of the inscribed circle.

Construction: Describe $\odot$ with given radius, any center.

Construct central $\angle =$ to given $\angle$. Draw two tangents $\parallel$ to these radii; etc.

V. Construct a **square**, having given:

42. One side.

43. The diagonal.

44. The perimeter.

45. The sum of a diagonal and a side.

VI. Construct a **rhombus**, having given:

46. One side and an angle adjoining it.

47. One side and the altitude.

48. The diagonals.

49. One side and one diagonal. [Use 269.]

50. An angle and the diagonal to the same vertex.

51. An angle and the diagonal between two other vertices.

52. One side and the radius of the inscribed circle.

VII. Construct a **rectangle**, having given:

53. Two adjoining sides.

54. A diagonal and a side.

55. One side and the angle formed by the diagonals.

56. A diagonal and the sum of two adjoining sides. [See Ex. 21.]

57. A diagonal and the perimeter.

58. The perimeter and the angle formed by the diagonals.

Construction: Bisect the perimeter and take $AB =$ to half of it. Bisect $\angle K$. At A construct $\angle BAX =$ to half $\angle K$. Etc.

VIII. Construct a **parallelogram**, having given:

59. One side and the diagonals.

60. The diagonals and the angle between them.

61. One side, an angle, and the diagonal not to the same vertex.

62. One side, an angle, and the diagonal to the same vertex.

63. One side, an angle, and the altitude upon that side.

64. Two adjoining sides and the altitude.

IX. Construct an **isosceles trapezoid**, having given

65. The bases and an angle adjoining the larger base.

66. The bases and an angle adjoining the less base.

67. The bases and the diagonal.

68. The bases and the altitude.

69. The bases and one of the equal sides.

70. One base, an angle adjoining it, and one of the equal sides.

71. One base, the altitude, and one of the equal sides.

72. One base, the radius of the circumscribed circle, and one of the equal sides. [First, describe a ⊙.]

73. One base, an angle adjoining it, and the radius of the circumscribed circle.

74. The bases and the radius of the circumscribed circle.

75. One base and the radius of the inscribed circle.

Construction: Bisect the base and erect a $\perp$ = to radius; etc.

X. Construct a **trapezoid**,* having given:

76. The bases and the angles adjoining one of them.

Construction: Take EC = to longer base, and on it take ED = to less base. Construct $\triangle DBC$ (by 271).

77. The four sides.

78. A base, the altitude, and the non-parallel sides.

Construction: Construct a $\triangle$ two sides of which equal the given non-$\parallel$ sides of the trapezoid, and the altitude from same vertex of which equals the given altitude. (See Ex. 34.)

79. The bases, an angle, and the altitude.

Construction: Construct $\square$ on ED, having given altitude and $\angle$.

80. A base, the angles adjoining it, and the altitude.

81. The longer base, an angle adjoining it, and the non-parallel sides.

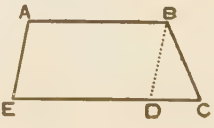
82. The shorter base, an angle not adjoining it, and the non-parallel sides.

XI. Construct the **locus** of a point which shall be:

83. At a given distance from a given point.

84. At a given distance from a given line.

* Note. It is evident that every trapezoid may be divided into a parallelogram and a triangle by drawing one line (as BD) $\parallel$ to one of the non-$\parallel$ sides. Hence the construction of a trapezoid is often merely constructing a triangle and a parallelogram.



85. At a given distance from a given circle:

(*i*) If the given radius is $<$ the given distance;
(*ii*) If the given radius is $>$ the given distance.

86. Equally distant from two given points.

87. Equally distant from two intersecting lines.

XII. Find (by intersecting loci) * the **point** P, which shall be:

88. At two given distances from two given points.†

89. Equally distant from three given points.

90. In a given line and equally distant from two given points.

91. In a given line and equally distant from two given intersecting lines.

92. In a given circle and equally distant from two given points.†

93. In a given circle and equally distant from two intersecting lines.†

94. Equally distant from two given intersecting lines and equally distant from two given points.†

95. At a given distance from a given line and equally distant from two given points.†

96. At a given distance from a given line and equally distant from two other intersecting lines.†

97. Equally distant from two given points and at a given distance from one of them.†

98. Equally distant from two given intersecting lines and at a given distance from one of them.†

99. At a given distance from a point and equally distant from two other points.†

100. At given distances from two given intersecting lines.†

101. At given distances from a given line and from a given circle.†

* It is well to draw the loci concerned as dotted lines. (See Ex. 105.)

† In the **Discussion**, include the answers to questions like these:
(1) Is this ever impossible? (*i.e.* must there always be such a point?)
(2) Are there ever two such points? when?
(3) Are there ever more than two? when?
(4) Is there ever only one? when?

102. At given distances from a given line and from a given point.*

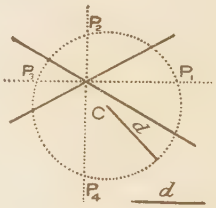
103. Equally distant from two parallels and equally distant from two intersecting lines.*

104. At a given distance from a given point and equally distant from two given parallels.*

105. At a given distance from a given point and equally distant from two given intersecting lines.

Can C be so taken that there will be no point?

Can C be so taken that there will be only one point? only two? only three? more than four?



XIII. Find (by intersecting loci) the **center** of a circle which shall:

106. Pass through three given points.†

107. Pass through a given point and touch a given line at a given point.†

108. Have a given radius and be tangent to a given line at a given point.†

109. Have a given radius, touch a given line, and pass through a given point.†

110. Pass through a given point and touch two given parallel lines.†

111. Touch two given parallels, one of them at a given point.†

112. Have a given radius and touch two given intersecting lines.†

113. Have a given radius and pass through two given points.†

114. Touch three given indefinite lines, no two of them being parallel.‡

115. Touch three given lines, only two of them being parallel.

XIV. Construct a **circle** which shall:

116. Pass through a given point and touch a given line at a given point.

117. Touch two given parallel lines, one of them at a given point.

118. Pass through a given point and touch two given parallels.

* See note (†) on preceding page.

† **Discussion**: Is this ever impossible? Are there ever two circles and hence two centers? Are there ever more than two? Etc.

‡ Four solutions. One is in 265.

119. Have a given radius, touch a given line, and pass through a given point.

120. Have its center in one line, touch another line, and have a given radius.

121. Have a given radius and touch two given intersecting lines.

122. Have a given radius and pass through two given points.

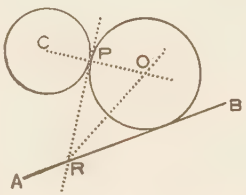
123. Have a given radius and touch a given circle at a given point. [Draw tangent to the given ⊙ at the given point.]

124. Have a given radius and touch two given circles.

125. Touch three indefinite intersecting lines.*

126. Touch two given intersecting lines, one of them at a given point.

127. Touch a given line and a given circle at a given point.



Given: Line AB; ⊙ C; point P.

Construction: Draw radius CP. Draw tangent at P meeting AB at R. Bisect $\angle PRB$, meeting CP produced at O; etc.

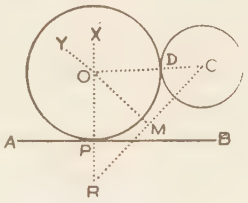
128. Be inscribed in a given sector.

Construction: Produce the radii to meet the tangent at the midpoint of the arc. In this △ inscribe a ⊙.

129. Have a given radius and touch two given intersecting circles.

130. Have a given radius, touch a given line, and a given circle.

131. Touch a given line at a given point and touch a given circle.



Given: Line AB; point P; ⊙ C.

Construction: At P erect $PX \perp$ to AB, and extend it below AB, so PR = radius ⊙ C.

Draw CR and bisect it at M.

Erect $MY \perp$ to CR at M, meeting PX at O; etc.

---

132. What is the locus of the vertices of all right triangles having the same hypotenuse?

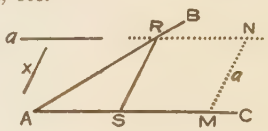
133. Through a given point on a given circumference draw two equal chords perpendicular to each other.

* Four solutions. One is in 265.

134. Draw a line of given length through a given point and terminating in two given parallels.

Construction: Use any point of one of the ||s as center and the given length as radius to describe an arc meeting the other ||. Join these two points. Through the given point draw a line ||, etc.

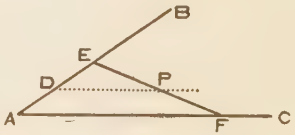
135. Draw a line, terminating in the sides of an angle, which shall be equal to one line and parallel to another.



Statement: $RS = a$, and is || to x.

136. Draw a line through a given point within an angle, which is terminated by the sides of the angle and bisected by the point.

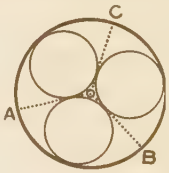
Construction: Through P draw PD || to AC. Take on AB, $DE = AD$. Draw EPF; etc.



137. Circumscribe a circle about a rectangle.

138. Construct three circles having the vertices of a given triangle as centers so that each touches the other two.

Construction: Inscribe a ⊙ in the △; etc.



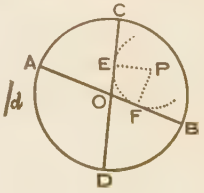
139. Construct within a circle three equal circles each of which touches the given circle and the other two.

Construction: Draw a radius, OA, and construct $\angle AOB =$ to 120° and $\angle AOC =$ to 120°. In these sectors inscribe, etc.

140. Through a point without a circle draw a secant having a given distance from the center.

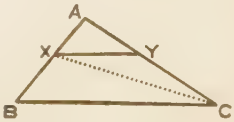
141. Draw a diameter to a circle at a given distance from a given point.

142. Through two given points within a circle draw two equal and parallel chords.



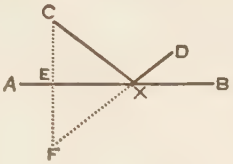
Construction: Bisect the line joining the given points and draw a diameter, etc.

143. Draw a parallel to side BC of triangle ABC, meeting AB in X and AC in Y, such that $XY = YC$.



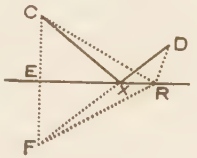
144. Find the locus of the points of contact of the tangents drawn to a series of concentric circles from an external point.

145. **Given:** Line AB and points C and D on the same side of it; find point X in AB such that $\angle AXC = \angle BXD$.



Construction: Draw $CE \perp$ to AB and produce to F so that $EF = CE$. Draw FD meeting AB in X. Draw CX.

146. Draw from one given point to another the shortest path which has one point in common with a given line.



Statement: $CX + XD$ is $< CR + RD$.

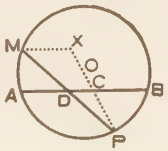
Another statement of this exercise: If C is an object before a mirror ER, and D is an eye, draw a diagram showing the path of a ray of light, from C, reflected to D.

147. Draw a line parallel to side BC of triangle ABC meeting AB at X and AC at Y, so that $XY = BX + YC$.

Construction: Draw bisectors of $\measuredangle B$ and C, meeting at O, etc.

148. Draw in a circle, through a given point of an arc, a chord that is bisected by the chord of the arc.

Construction: Draw radius OP meeting chord at C. Prolong PO to X so that $CX = CP$. Draw $XM \parallel$ to AB meeting $\odot$ at M. Draw PM cutting AB at D; etc. Is there any other chord from P bisected by AB?



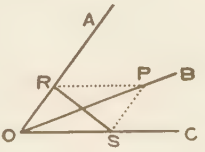
149. Inscribe in a given circle a triangle whose angles are given.

Construction: Construct 3 central $\measuredangle$s, doubles of the given $\measuredangle$s. Etc.

150. Circumscribe about a given circle a triangle whose angles are given.

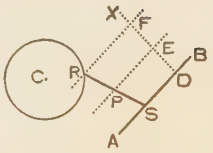
Construction: Inscribe $\triangle$ (like Ex. 149) first, and draw tangents $\parallel$ to the sides.

151. Three lines meet in a point. Draw a line terminating in the outer two and bisected by the inner one.



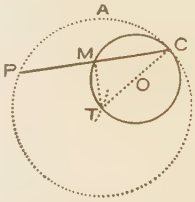
Construction: Through any point P, of OB, draw $\parallel$s to the outer lines. Draw diagonal RS; etc.

152. Draw through a given point P, a line that is terminated by a given circle and a given line and is bisected by P.



Construction: Draw any line DX meeting AB at D. Draw $PE \parallel$ to AB meeting DX at E. Take $EF = ED$; etc.

153. Through a given point without a circle draw a secant to the circle which is bisected by the circle.

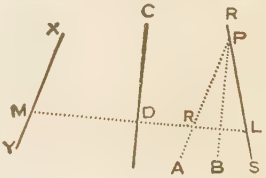


Construction: Draw arc at T, using P as center and diam. of $\odot O$ as radius. Using T as center and same radius as before, describe circle touching $\odot O$ at C and passing through P. Draw PC meeting $\odot O$ at M.

154. Inscribe a square in a given rhombus.

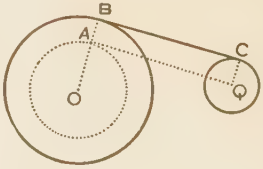
[Bisect the four $\measuredangle$ formed by the diagonals.]

155. Bisect the angle formed by two lines without producing them to their point of intersection.



Construction: At P, any point in RS, draw $PA \parallel$ to XY; bisect $\angle APS$ by PB. At any point in PB erect $ML \perp$ to PB, meeting the given lines in M and L. Bisect ML at D and erect $DC \perp ML$, etc.

156. Construct a common external tangent to two circles.



Construction: Using O as a center and a radius equal to the difference of the given radii, construct (dotted) circle. Draw QA tangent to this $\odot$ from Q; draw radius OA and produce it to meet given $\odot$ at B. Draw radius $QC \parallel$ to OB. Join BC.

Statement: BC is tangent to both Ⓢ.

Proof: $AB = CQ$ (Const.). AB is $\parallel$ to CQ (?).

$\therefore ABCQ$ is a $\square$ (?). But $\angle OAQ$ is a rt. $\angle$; etc.

157. Construct a common internal tangent to two circles.

Construction: Using O as a center and a radius equal to the sum of the given radii, construct (dotted) circle. Draw QA tangent to this $\odot$ from Q. Draw radius OA meeting given $\odot$ at B, etc., as above.

BOOK III

PROPORTION. SIMILAR FIGURES

274. A **ratio** is the quotient of one quantity divided by another, both being of the same kind.

275. A **proportion** is an equation whose members are ratios.

276. The **extremes** of a proportion are the first and the last terms. The **means** of a proportion are the second and the third terms.

277. The **antecedents** are the first and the third terms. The **consequents** are the second and the fourth terms.

278. A **mean proportional** is the second or the third term of a proportion in which the means are identical.

A **third proportional** is the last term of a proportion in which the means are identical.

A **fourth proportional** is the last term of a proportion in which the means are not identical.

279. A **series of equal ratios** is the equality of more than two ratios.

A **continued proportion** is a series of equal ratios in which the consequent of any ratio is the antecedent of the next following ratio.

EXPLANATORY. A ratio is written as a fraction or as an indicated division; $\frac{a}{b}$, or $a \div b$, or $a : b$. A proportion is usually written $\frac{a}{b} = \frac{x}{y}$, or $a : b = x : y$, and is read: "a is to b as x is to y." In this proportion the extremes are a and y; the means are b and x; the antecedents are a and x; the consequents are b and y; and y is a fourth proportional to a, b, x. In the proportion $a : m = m : z$, the mean proportional is m, and z is the third proportional.

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

280. In a proportion the product of the extremes is equal to the product of the means.

Given: $\frac{a}{b} = \frac{x}{y}$ or $a : b = x : y$. **To Prove:** $ay = bx$.

Proof: $\frac{a}{b} = \frac{x}{y}$ (Hyp.). Multiply by the common denominator by and obtain, $ay = bx$ (Ax. 3).

Q.E.D.

Proposition II. Theorem

281. If the product of two quantities is equal to the product of two others, one pair may be made the extremes of a proportion and the other pair the means.

Given: $ay = bx$.

To Prove: These eight proportions:

1. $a : b = x : y$,
2. $a : x = b : y$,
3. $b : a = y : x$,
4. $b : y = a : x$,
5. $x : y = a : b$,
6. $x : a = y : b$,
7. $y : x = b : a$,
8. $y : b = x : a$.

Proof. 1. $ay = bx$ (Hyp.).

Divide each member by by, $\frac{ay}{by} = \frac{bx}{by}$ (Ax. 3).

$\therefore \frac{a}{b} = \frac{x}{y}$ or $a : b = x : y$ Q.E.D.

2. $ay = bx$ (Hyp.).

Divide by xy, etc.

3. $ay = bx$ (?).

Divide by ax, etc. Etc., etc.

Numerical Illustration. Suppose in this paragraph $a = 4, b = 14$, $x = 6$, $y = 21$; the truth of the above proportions can be clearly seen by writing these equivalents. $4 \times 21 = 14 \times 6$ (True).

1. $4 : 14 = 6 : 21$ (True); 2. $4 : 6 = 14 : 21$ (True); etc.

They will all be recognized as true proportions.

Proposition III. Theorem

282. In any proportion the terms are also in proportion by alternation (that is, the first term is to the third as the second is to the fourth).

Given: $a:b=x:y$. **To Prove:** $a:x=b:y$.

Proof:

$a:b=x:y$ (Hyp.).

$\therefore ay=bx$ (280).

$\therefore a:x=b:y$ (281).

Q.E.D.

Proposition IV. Theorem

283. In any proportion the terms are also in proportion by inversion (that is, the second term is to the first as the fourth term is to the third).

[The proof is similar to the proof of 282.]

Proposition V. Theorem

284. In any proportion the terms are also in proportion by composition (that is, the sum of the first two terms is to the first, or the second, as the sum of the last two terms is to the third, or the fourth).

Given: $a:b=x:y$. **To Prove:** $\begin{cases} a+b:a=x+y:x, \text{ or} \\ a+b:b=x+y:y. \end{cases}$

Proof: $a:b=x:y$ (Hyp.). $\therefore ay=bx$ (280).

Add ax to each, and obtain, $ax+ay=ax+bx$ (Ax. 2).

That is, $a(x+y)=x(a+b)$.

Hence $a+b:a=x+y:x$ (281).

Similarly, by adding by, $a+b:b=x+y:y$. Q.E.D.

Ex. 1. Is the equation $12:9=28:21$ a true proportion?

Ex. 2. Apply Proposition I to the above proportion.

Ex. 3. Apply Proposition III to the above proportion.

Ex. 4. Apply Proposition IV to the above proportion.

Ex. 5. Apply Proposition V to the above proportion.

Proposition VI. Theorem

285. In any proportion the terms are also in proportion by division (that is, the difference between the first two terms is to the first, or the second, as the difference between the last two terms is to the third, or the fourth).

Given: $a : b = x : y$. **To Prove:** $\begin{cases} a-b : a = x-y : x, \text{ or} \\ a-b : b = x-y : y. \end{cases}$

Proof: $a : b = x : y$ (Hyp.). $\therefore ay = bx$ (280).

Subtracting each side from ax, $ax - ay = ax - bx$ (Ax. 2).

That is, $a(x-y) = x(a-b)$.

Hence $a-b : a = x-y : x$ (281).

Similarly, $a-b : b = x-y : y$. Q.E.D.

Note 1. The proportions of 284 and 285 may be written in many different forms (282, 283). Thus, (1) $a \pm b : a = x \pm y : x$;

(2) $a \pm b : b = x \pm y : y$; (3) $a \pm b : x \pm y = a : x$, etc.

Note 2. In any proportion the sum of the antecedents is to the sum of the consequents as either antecedent is to its consequent. Also, in any proportion the difference of the antecedents is to the difference of the consequents as either antecedent is to its consequent. (Explain.)

Thus: $a + x : b + y = a : b = x : y$. Also, $a - x : b - y = a : b = x : y$.

Proposition VII. Theorem

286. In any proportion the terms are also in proportion by composition and division (that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference).

Given: $a : b = x : y$. **To Prove:** $\dfrac{a+b}{a-b} = \dfrac{x+y}{x-y}$.

Proof:

$$\frac{a+b}{a} = \frac{x+y}{x} \quad (284).$$

$$\frac{a-b}{a} = \frac{x-y}{x} \quad (285).$$

Divide the first by the second, $\dfrac{a+b}{a-b} = \dfrac{x+y}{x-y}$ (Ax. 3).

Q.E.D.

Proposition VIII. Theorem

287. In any proportion, like powers of the terms are also in proportion, and like roots of the terms are in proportion.

Given: $a : b = x : y$.

To Prove: $a^n : b^n = x^n : y^n$; and $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{x} : \sqrt[n]{y}$.

Proof: [Write the hypothesis in fractional form, etc.]

Proposition IX. Theorem

288. In two or more proportions the products of the corresponding terms are also in proportion.

Given: $a : b = x : y$, and $c : d = l : m$, and $e : f = r : s$.

To Prove: $ace : bdf = xlr : yms$.

Proof: [Write in fractional form and multiply.]

Proposition X. Theorem

289. A mean proportional is equal to the square root of the product of the extremes.

Given: $a : x = x : b$. **To Prove:** $x = \sqrt{ab}$.

Proof: [Use 280.]

Proposition XI. Theorem

290. If three terms of one proportion are equal to the corresponding three terms of another proportion, each to each, the remaining terms are also equal.

Given: $\left\{\begin{matrix} a : b = c : m, \text{ and} \\ a : b = c : r. \end{matrix}\right\}$. **To Prove:** $m = r$.

Proof:

$am = bc$ and $ar = bc$ (280).

$\therefore am = ar$ (Ax. 1).

$\therefore m = r$ (Ax. 3).

Q.E.D.

Proposition XII. Theorem

291. In a series of equal ratios the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.

Given: $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$.

To Prove: $\frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=$ etc.

Proof: Set each given ratio = to m; thus,

$$\frac{a}{b}=m;\ \frac{c}{d}=m;\ \frac{e}{f}=m;\ \frac{g}{h}=m.$$

$$\therefore a=bm,\ c=dm,\ e=fm,\ g=hm \qquad \text{(Ax. 3).}$$

Hence, $\frac{a+c+e+g}{b+d+f+h}=\frac{bm+dm+fm+hm}{b+d+f+h}$ (Substitution).

$$=\frac{m(b+d+f+h)}{b+d+f+h} \qquad \text{(Factoring).}$$

$$=m \qquad \text{(Canceling).}$$

$$\therefore \frac{a+c+e+g}{b+d+f+h}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h} \qquad \text{(Ax. 1).}$$

Q.E.D.

Ex. 1. If $3:4=6:x$, find x. **Ex. 2.** If $8:12=12:x$, find x.

Ex. 3. Find a fourth proportional to 6, 7, and 15.

Ex. 4. Find a third proportional to 4 and 10.

Ex. 5. If $11:15=x:25$, find x. **Ex. 6.** If $4:x=x:25$, find x.

Ex. 7. Find a mean proportional between 8 and 18.

Ex. 8. If $7:x=35:48$, find x.

Ex. 9. Given, $5:8=15:24$. Write seven other true proportions containing these four numbers.

Ex. 10. If $5\times 6=2\times 15$, write eight proportions with these numbers.

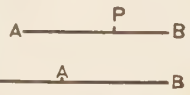
Ex. 11. If $7:12=21:36$, write the proportion resulting by alternation; inversion; composition; division; composition and division.

Ex. 12. If $x+y:x-y=17:7$, write the proportions that result by virtue of composition; division; composition and division.

Ex. 13. Apply 291 to the ratios, $\frac{2}{3}=\frac{4}{6}=\frac{6}{9}=\frac{8}{12}=\frac{10}{15}$.

292. **A segment** of a line is any part of the line. If a line is divided by a point into two segments, they are the distances from this point of division to the extremities of the line

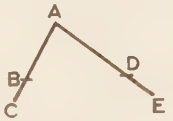
The upper line AB is divided **internally** by P (a point between the extremities A and B) into segments AP and PB. The lower line AB is divided **externally** by R (a point in the prolongation of AB) into segments, RA and RB.



When a line is divided *internally*, it equals the *sum* of the segments; when it is divided *externally*, it equals the difference of the segments.

Two lines are **divided proportionally** if the ratio of the lines is equal to the ratio of the corresponding segments. Thus, AC and AE are said to be "divided proportionally" if

$$\frac{AC}{AE}=\frac{AB}{AD} \text{ or } \frac{AC}{AE}=\frac{BC}{DE} \text{ or } \frac{AC}{AE}=\frac{AB}{AD}=\frac{BC}{DE}.$$



NOTE. We have seen that it is possible to add two lines and subtract one line from another. Now it is essential that we clearly understand the significance implied by indicating the multiplication or the division of one line by another.

What is actually done is to multiply or divide the numerical measure of one line by the numerical measure of another. Thus, if one line is 8 inches long and another is 18 inches long, we say that the ratio of the first line to the second is $\frac{8}{18}$ or $\frac{4}{9}$, meaning that the smaller line is four ninths of the larger.

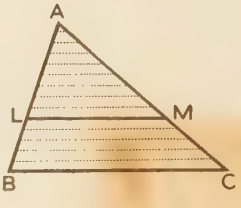
Also, in referring to the product of two lines, we merely understand that the product of their numerical measures is intended.

If a line is multiplied by itself, we obtain the square of the numerical measure of the line. The square of the line AB is written AB^2 or $(AB)^2$, and the quantity that is squared is the numerical value of the length of AB.

In the preceding paragraphs of Book III, we have been considering numerical magnitudes. It should be distinctly understood that in the following geometrical propositions and demonstrations, the foregoing interpretation is implied in multiplication and division involving lines.

Historical Note. Eudoxus, one of the most prominent of the Greek mathematicians, was famous also as a physician in the fourth century B.C. One of his principal contributions to geometry was the perfection of a rigorous theory of ratio and proportion.

PROPOSITION XIII. THEOREM



293. A line parallel to one side of a triangle divides the other sides into proportional segments.

Given: $\triangle ABC$ and line $LM \parallel$ to BC.

To Prove: $AL : LB = AM : MC$.

Proof: I. If the parts AL and LB are **commensurable.**

There exists a common unit of measure of AL and LB (225).

Suppose this is contained 9 times in AL and 5 times in LB.

Then $$\frac{AL}{LB} = \frac{9}{5} \qquad \text{(Ax. 3).}$$

Draw lines through the several points of division $\parallel$ to BC.

These divide AM into 9 parts and MC into 5 parts.

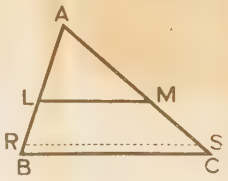
All these 14 parts are equal (140).

$$\therefore \frac{AM}{MC} = \frac{9}{5} \qquad \text{(Ax. 3).}$$

$$\therefore \frac{AL}{LB} = \frac{AM}{MC} \qquad \text{(Ax. 1).}$$

Q.E.D.

II. If the parts AL and LB are **incommensurable.**



There does not exist a common unit (225).

Divide AL into several equal parts (by 261).

Apply one of these as a unit of measure to LB. There is remainder, RB (225).

Draw $RS \parallel$ to BC.

Now $$\frac{AL}{LR} = \frac{AM}{MS} \qquad \text{(Case I).}$$

Indefinitely increase the *number* of equal parts of AL. That is, indefinitely decrease each part, the unit or divisor. Hence, the remainder, RB, is indefinitely decreased. (Because the remainder is $<$ the divisor.)

That is, RB approaches zero as a limit.

Also SC approaches zero as a limit.

$\therefore$ LR approaches LB as a limit (227).

Also MS approaches MC as a limit (227).

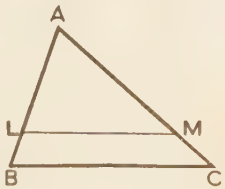
$\therefore \frac{AL}{LR}$ approaches $\frac{AL}{LB}$ as a limit.

Also $\frac{AM}{MS}$ approaches $\frac{AM}{MC}$ as a limit.

Consequently, $\frac{AL}{LB} = \frac{AM}{MC}$ (229). Q.E.D.

Proposition XIV. Theorem

294. **If a line parallel to one side of a triangle intersects the other sides, it divides these sides proportionally.**



Given: $\triangle ABC$; $LM \parallel$ to BC.

To Prove:

I. $AB : AC = AL : AM$.

II. $AB : AC = LB : MC$.

Proof:

$AL : LB = AM : MC$ (293).

$\therefore AL + LB : AL = AM + MC : AM$ (284).

Also $AL + LB : LB = AM + MC : MC$ (284).

But $AL + LB = AB$, and $AM + MC = AC$ (Ax. 4).

Substituting, in last two proportions:

$AB : AL = AC : AM$ (Ax. 6).

Also $AB : LB = AC : MC$ (Ax. 6).

$\therefore$ I. $AB : AC = AL : AM$ (282).

Also II. $AB : AC = LB : MC$ (282). Q.E.D.

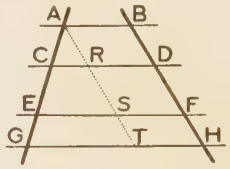
These proportions may be combined thus: $\frac{AB}{AC} = \frac{AL}{AM} = \frac{LB}{MC}$.

Each of the above proportions may be written in eight different ways.

Ex. If, in figure of 294, AB is 9 units, AC 12 units, and AL 6 units, find AM, LB, and MC.

Proposition XV. Theorem

295. Three or more parallels intercept proportional segments on two transversals.



Given: (?).

To Prove:

$AC : BD = CE : DF = EG : FH$.

Proof: Draw from A, $AT \parallel$ to BH intersecting the ||s, etc.

In $\triangle AES$, $\quad \frac{AE}{AS} = \frac{AC}{AR} = \frac{CE}{RS}$ (294).

In $\triangle AGT$, $\quad \frac{AE}{AS} = \frac{EG}{ST}$ (293).

$\therefore \frac{AC}{AR} = \frac{CE}{RS} = \frac{EG}{ST}$ (Ax. 1).

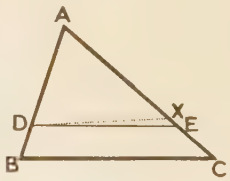
But $\quad AR = BD,\ RS = DF,\ ST = FH$ (124).

Hence $\quad AC : BD = CE : DF = EG : FH$ (Ax. 6).

Q.E.D.

Proposition XVI. Theorem

296. If a line divides two sides of a triangle proportionally, it is parallel to the third side.



Given: $\triangle ABC$; line DE; the proportion $AB : AC = AD : AE$.

To Prove: DE is $\parallel$ to BC.

Proof: Through D draw $DX \parallel$ to BC meeting AC at X.

$\therefore AB : AC = AD : AX$ (294).

But $\quad AB : AC = AD : AE$ (Hyp.).

$\therefore AX = AE$ (290).

$\therefore DX$ and DE coincide (39).

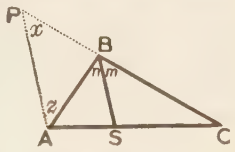
That is, $\quad DE$ is $\parallel$ to BC Q.E.D.

Ex. 1. Prove, as 296 is proved, that the line bisecting two sides of a triangle is parallel to the third side.

Ex. 2. In fig. of Proposition XIV, if AL is $\frac{3}{4}$ of AB, what is true of AM? of MC?

Proposition XVII. Theorem

297. The bisector of an angle of a triangle divides the opposite side into segments that are proportional to the other two sides.



Given: $\triangle ABC$; BS the bisector of $\angle ABC$.

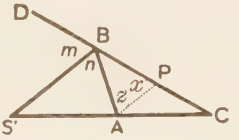
To Prove: $AS : SC = AB : BC$.

Proof: Through A draw $AP \parallel$ to BS, meeting CB, produced, at P.

Then, in $\triangle PAC$,	$AS : SC = PB : BC$	(293).
Now	$\angle m = \angle x$	(67).
And	$\angle n = \angle z$	(66).
But	$\angle m = \angle n$	(Hyp.).
	$\therefore \angle x = \angle z$	(Ax. 1).
	$\therefore PB = AB$	(114).
Substituting above,	$AS : SC = AB : BC$	(Ax. 6). Q.E.D.

Proposition XVIII. Theorem

298. The bisector of an exterior angle of a triangle divides the opposite side (externally) into segments that are proportional to the other two sides.



Given: $\triangle ABC$; BS', the bisector of exterior $\angle ABD$, meeting AC (externally) at S'.

To Prove: $S'A : S'C = AB : BC$.

Proof: Through A draw $AP \parallel$ to BS' meeting BC at P.

Then, in $\triangle CBS'$,	$S'A : S'C = BP : BC$	(294).
Now	$\angle m = \angle x$	(67).
And	$\angle n = \angle z$	(66).
But	$\angle m = \angle n$	(Hyp.).
	$\therefore \angle x = \angle z$	(Ax. 1).
	$\therefore BP = AB$	(114).
Substituting above,	$S'A : S'C = AB : BC$	(Ax. 6). Q.E.D.

Ex. 1. If, in 297, $AS = 3$, $AB = 4$, $BC = 9$, find SC.

Ex. 2. If, in 297, $AC = 20$, $AB = 9$, $BC = 21$, find AS and SC.

Ex. 3. If, in 298, $S'A = 10$, $AB = 7$, $BC = 16$, find $S'C$ and AC.

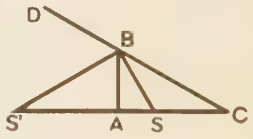
Ex. 4. If, in 298, $AC = 14$, $AB = 12$, $BC = 19$, find $S'A$ and $S'C$.

299. A line is **divided harmonically** if it is divided internally and externally in the same ratio.

In 297, the line AC is divided *internally* by S, in the ratio $AB : BC$.
In 298, the line AC is divided *externally* by S', in the ratio $AB : BC$.

Proposition XIX. Theorem

300. The bisectors of the interior and exterior angles of a triangle (at a vertex) divide the opposite side harmonically.



Given: $\triangle ABC$; BS bisecting $\angle ABC$; and BS' bisecting $\angle ABD$.

To Prove: $AS : SC = S'A : S'C$.

Proof:

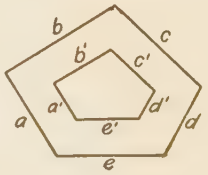
$$\frac{AS}{SC} = \frac{AB}{BC} \qquad (297).$$

$$\frac{S'A}{S'C} = \frac{AB}{BC} \qquad (298).$$

$$\therefore \frac{AS}{SC} = \frac{S'A}{S'C} \qquad (\text{Ax. } 1).$$

Q.E.D.

301. Similar polygons are polygons that are mutually equiangular and the homologous sides of which are proportional. That is, every pair of homologous angles are equal; and the ratio of one pair of homologous sides is equal to the ratio of every other pair of homologous sides,



$$a : a' = b : b' = c : c' = d : d' = \text{etc.}$$

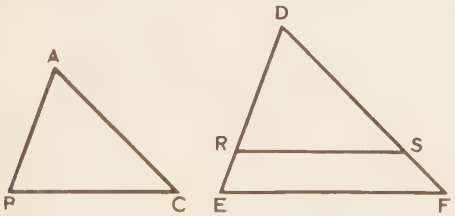
Triangles are **similar** if they are mutually equiangular and their homologous sides are proportional.

Proposition XX. Theorem

302. Two triangles are similar if they are mutually equiangular.

Given: $\triangle ABC$, DEF; $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

To Prove: The $\triangle$ are similar (that is, their sides are proportional).



Proof: Place $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and $\triangle ABC$ takes the position of $\triangle DRS$.

Then $\angle DRS = \angle E$ (Hyp.).

$\therefore RS$ is $\parallel$ to EF (71).

$\therefore DE : DR = DF : DS$ (294).

That is, $DE : AB = DF : AC$ (Ax. 6).

Likewise, by placing $\triangle ABC$ upon $\triangle DEF$ so that $\angle B$ coincides with its equal, $\angle E$, we may prove that

$DE : AB = EF : BC$

$\therefore DE : AB = DF : AC = EF : BC$ (Ax. 1).

$\therefore$ the $\triangle$ are similar (301).

Q.E.D.

303. Corollary. Two triangles are similar if two angles of one are equal to two angles of the other. (111 and 302.)

304. Corollary. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other. (303.)

Ex. 1. If two transversals intersect between two parallels, the triangles formed are similar.

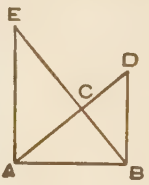
Ex. 2. Two isosceles triangles are similar if a base angle of one is equal to a base angle of the other.

Ex. 3. Two isosceles triangles are similar if the vertex angle of one is equal to the vertex angle of the other.

Ex. 4. The line joining the midpoints of two sides of a triangle forms a triangle similar to the original triangle.

Ex. 5. The diagonals of a trapezoid form, with the parallel sides, two similar triangles.

Ex. 6. If at the extremities of the hypotenuse of a right triangle perpendiculars are erected meeting the legs produced, the new triangles formed are similar.



Ex. 7. In the figure of Ex. 6, prove:

(1) Triangle ABC similar to each of the triangles ACE and BCD.

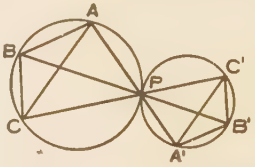
(2) Triangle ABE similar to triangle ABD.

(3) Triangle ACE similar to triangle ABD.

(4) Triangle BCD similar to triangle ABE.

(5) Triangles ABC, ABD, ABE similar.

Ex. 8. Two circles are tangent externally at P; through P three lines are drawn, meeting one circumference in A, B, C, and the other in A', B', C'. The triangles ABC and $A'B'C'$ are similar.



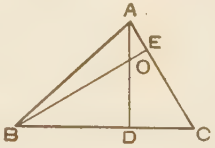
Ex. 9. Prove the same theorem if the circles are tangent internally.

Ex. 10. If two circles are tangent externally at P, and BB', CC' are drawn through P, terminating in the circumferences, the triangles PBC and $PB'C'$ are similar.

[Draw the common tangent at P.]

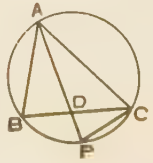
Ex. 11. Prove the same theorem if the circles are tangent internally.

Ex. 12. If AD and BE are two altitudes of triangle ABC, the triangles ACD and BCE are similar.



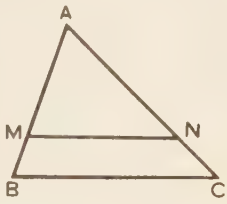
Ex. 13. If AD and BE are two altitudes of triangle ABC, meeting at O, the triangles BOD and AOE are similar.

Ex. 14. Triangle ABC is inscribed in a circle and AP is drawn to P, the midpoint of arc BC, meeting chord CB at D. The triangles ABD and ACP are similar.



Proposition XXI. Theorem

305. If a line parallel to one side of a triangle intersects the other sides, the triangle formed is similar to the original triangle.



Given: $MN \parallel$ to BC in $\triangle ABC$.

To Prove: $\triangle AMN$ similar to $\triangle ABC$.

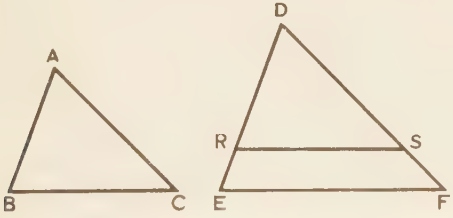
Proof: In the $\triangle AMN$ and ABC,

$\angle A = \angle A$	(Ident.).
$\angle AMN = \angle B$	(67).
$\therefore \triangle AMN$ is similar to $\triangle ABC$	(303).

Q.E.D.

Proposition XXII. Theorem

306. If two triangles have an angle of one equal to an angle of the other and the sides including these angles proportional, the triangles are similar.



Given: $\triangle ABC$ and DEF; $\angle A = \angle D$; $DE : AB = DF : AC$.

To Prove: The $\triangle$ similar.

Proof: Superpose $\triangle ABC$ upon $\triangle DEF$ so that $\angle A$ coincides with its equal, $\angle D$, and $\triangle ABC$ takes the position of $\triangle DRS$.

Now $DE : DR = DF : DS$	(Hyp.).
$\therefore RS$ is $\parallel$ to EF	(296).
$\therefore \triangle DRS$ is similar to $\triangle DEF$	(305).
Substituting, $\triangle ABC$ is similar to $\triangle DEF$	(Ax. 6).

Q.E.D.

307. Corollary. If two triangles are similar to the same triangle, they are similar to each other.

Proof: The three angles of each of the first two triangles are respectively equal to the three angles of the third (301).

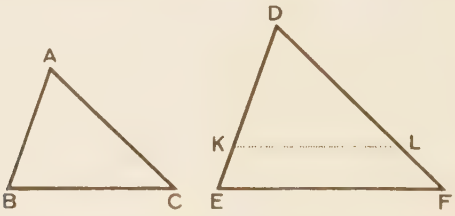
Hence the first two ◬ are mutually equiangular (Ax. 1).

Therefore they are similar (302).

Q.E.D.

Proposition XXIII. Theorem

308. If two triangles have their homologous sides proportional, they are similar.



Given: ◬ ABC and DEF; $DE : AB = DF : AC = EF : BC$.

To Prove: △ ABC similar to △ DEF.

Proof: On DE take DK = to AB; and on DF take DL = to AC. Draw KL.

1st Now	$DE : AB = DF : AC$	(Hyp.).
	$DE : DK = DF : DL$	(Ax. 6).
	$\therefore$ KL is ∥ to EF	(296).
	△ DKL is similar to △ DEF	(305).
2d	$DE : DK = EF : KL$ (Def. sim. ◬)	(301).
Substituting,	$DE : AB = EF : KL$	(Ax. 6).
But	$DE : AB = EF : BC$	(Hyp.).
	$\therefore$ $BC = KL$	(290).
3d	△ ABC is congruent to △ DKL	(78).

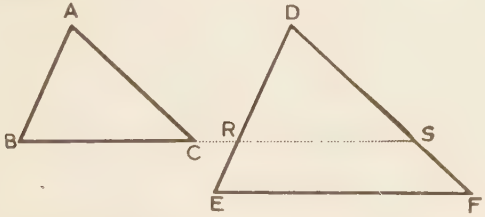
But △ DKL has been proved similar to △ DEF.

Substituting, △ ABC is similar to △ DEF (Ax. 6).

Q.E.D.

Proposition XXIV. Theorem

309. If two triangles have their homologous sides parallel, they are similar.



Given: $\triangle ABC$ and DEF; $AB \parallel$ to DE; $AC \parallel$ to DF; and $BC \parallel$ to EF.

To Prove: $\triangle ABC$ similar to $\triangle DEF$.

Proof: Produce BC of $\triangle ABC$ until it intersects two sides of $\triangle DEF$ at R and S.

Now $\angle B = \angle DRS$, and $\angle DRS = \angle E$ (67).

$\therefore \angle B = \angle E$ (Ax. 1).

Similarly, we may prove $\angle ACB =$ to $\angle F$.

$\therefore \triangle ABC$ is similar to $\triangle DEF$ (303).

Q.E.D.

Ex. 1. Draw the figure and prove Proposition XXIV:

(*a*) If one triangle is entirely within the other;

(*b*) If no side of either, when prolonged, meets any side of the other;

(*c*) If one side of one intersects two sides of the other, without prolongation;

(*d*) If a vertex of one is upon a side of the other.

Ex. 2. If in a right triangle, a perpendicular is drawn from the vertex of the right angle upon the hypotenuse, the two new right triangles are similar to the original triangle and to each other.

Ex. 3. Are all equilateral triangles similar? Why?

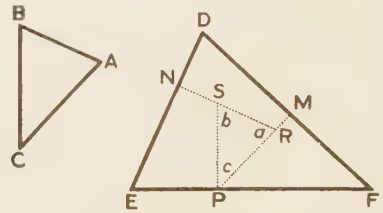
Ex. 4. Are all squares similar? all rectangles? Why?

Ex. 5. Do a square and a rectangle fulfill either of the conditions for similar polygons?

Ex. 6. If from any point in a leg of a right triangle a line is drawn perpendicular to the hypotenuse, the triangles are similar.

Proposition XXV. Theorem

310. If two triangles have their homologous sides perpendicular, they are similar.



Given: $\triangle ABC$ and DEF; $AB \perp$ to DE; $AC \perp$ to DF; etc.

To Prove: $\triangle ABC$ similar to $\triangle DEF$.

Proof: Through P, any point in EF, construct $PR \parallel$ to AC, meeting DF at M. At R, any point in PM, draw $RS \parallel$ to AB, meeting ED at N. Draw $PS \parallel$ to BC, meeting NR at S, forming the $\triangle PRS$. Now PM is $\perp$ to DF and RN is $\perp$ to DE (64).

In quadrilateral $DMRN$,

$$\angle D + \angle M + \angle MRN + \angle N = 4 \text{ rt. } \measuredangle \quad (156).$$

But $$\angle M + \angle N = 2 \text{ rt. } \measuredangle \quad (16).$$

$$\therefore \angle D + \angle MRN = 2 \text{ rt. } \measuredangle \quad (\text{Ax. } 2).$$

But $$\angle a + \angle MRN = 2 \text{ rt. } \measuredangle \quad (46).$$

$$\therefore \angle D = \angle a \quad (49).$$

Similarly, by quadrilateral $EPSN$, it may be proved that

$$\angle E = \angle b.$$

$$\therefore \triangle DEF \text{ is similar to } \triangle PRS \quad (303).$$

But $$\triangle ABC \text{ is similar to } \triangle PRS \quad (309).$$

$$\therefore \triangle ABC \text{ is similar to } \triangle DEF \quad (307).$$

Q. E. D.

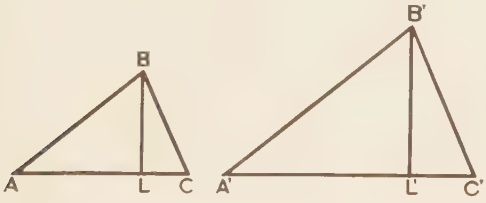
Historical Note. Thales, as early as the sixth century B.C., used similar triangles to determine the height of the great Egyptian pyramid. He measured the shadow of a pole of known height, and the shadow of the pyramid at the same time, to obtain homologous sides of similar right triangles.

Proposition XXVI. Theorem

311. Two homologous altitudes of two similar triangles are proportional to any two homologous sides.

Given: (?).

To Prove: $\frac{BL}{B'L'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$.



Proof: $\triangle ABC$ is similar to $\triangle A'B'C'$ (Hyp.).

$\therefore \angle A = \angle A'$ (301).

$\therefore \triangle ABL$ is similar to $\triangle A'B'L'$ (304).

$\therefore \frac{BL}{B'L'} = \frac{AB}{A'B'}$ (301).

But $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ (301).

$\therefore \frac{BL}{B'L'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ (Ax. 1).

Q.E.D.

312. In similar figures, homologous angles are equal (Def.).

313. In similar figures, homologous sides are proportional (Def.).

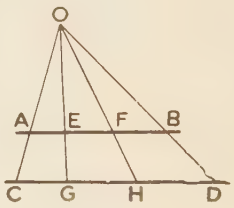
The antecedents of this proportion belong to one of the similar figures, and the consequents to the other.

314. In similar triangles, homologous sides are opposite homologous angles.

Shortest sides are homologous. (Opposite smallest ∡).
Medium sides are homologous. (Opposite medium ∡).
Longest sides are homologous. (Opposite largest ∡).

Proposition XXVII. Theorem

315. If two parallel lines are cut by three or more transversals that meet at a point, the corresponding segments of the parallels are proportional.



Given: (?).

To Prove: $\frac{AE}{CG} = \frac{EF}{GH} = \frac{FB}{HD}$.

Proof: In $\triangle OCG$, AE is $\parallel$ to CG (Hyp.).

$\therefore \triangle OAE$ is similar to $\triangle OCG$ (305).

Likewise, $\triangle OEF$ is similar to $\triangle OGH$, and $\triangle OFB$ is similar to $\triangle OHD$ (305).

$$\therefore \frac{AE}{CG} = \frac{OE}{OG}, \text{ and } \frac{EF}{GH} = \frac{OE}{OG} \quad (313).$$

$$\therefore \frac{AE}{CG} = \frac{EF}{GH} \quad (\text{Ax. } 1).$$

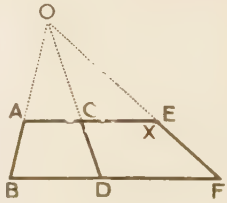
Likewise $$\frac{EF}{GH} = \frac{OF}{OH}, \text{ and } \frac{OF}{OH} = \frac{FB}{HD} \quad (313).$$

$$\therefore \frac{AE}{CG} = \frac{EF}{GH} = \frac{FB}{HD} \quad (\text{Ax. } 1).$$

Q.E.D.

Proposition XXVIII. Theorem

316. If three or more non-parallel transversals intercept proportional segments on two parallels, they meet at a point. [Converse.]



Given: Transversals AB, CD, EF; parallels AE and BF; proportion, $AC : BD = CE : DF$.

To Prove: AB, CD, EF meet at a point.

Proof: Produce AB and CD until they meet, at O.

Draw OF cutting AE at X.

Now $AC : BD = CX : DF$ (315).

But $AC : BD = CE : DF$ (Hyp.).

$\therefore CX = CE$ (290).

$\therefore FE$ and FX coincide (39).

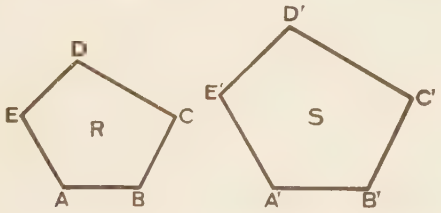
That is, FE prolonged, passes through O. Q.E.D.

Ex. 1. Show the truth of Proposition XXVIII. by an accurate diagram.

Ex. 2. In the figure of Proposition XXVIII, what is point X?

Proposition XXIX. Theorem

317. The perimeters of two similar polygons are to each other as any two homologous sides.



Given: Polygon R whose perimeter $= P$ and similar polygon S whose perimeter $= P'$.

To Prove: $P : P' = AB : A'B' = BC : B'C' = \text{etc.}$

Proof: $AB : A'B' = BC : B'C' = CD : C'D' = \text{etc.}$ (313).

$$\therefore \frac{AB + BC + CD + \cdots}{A'B' + B'C' + C'D' + \cdots} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \cdots \text{ etc.} \quad (291).$$

Substituting, $$\frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \text{etc.} \quad (\text{Ax. } 6).$$

Q.E.D.

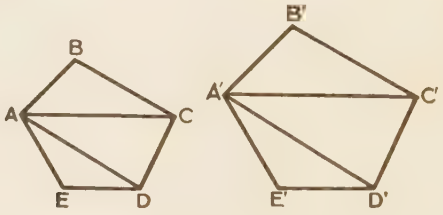
Ex. 1. The sides of a certain polygon are 4, 10, 12, 15, and 18. The shortest side of a similar polygon is 6. Find the other four sides.

Ex. 2. The perimeters of two similar polygons are 125 and 275 respectively. The longest side of the smaller polygon is 40. Find the longest side of the larger polygon.

Ex. 3. Enumerate the ways of proving two triangles similar. Which is the easiest of these ways?

Proposition XXX. Theorem

318. If two polygons are similar, they may be decomposed into the same number of triangles, similar each to each and similarly placed.



Given: Similar polygons BE and $B'E'$.

To Prove: $\triangle ABC$ similar to $\triangle A'B'C'$;
$\triangle ACD$ similar to $\triangle A'C'D'$;
$\triangle AED$ similar to $\triangle A'E'D'$.

Proof: First. $AB : A'B' = BC : B'C'$ (313).
Also $\angle B = \angle B'$ (312).
Therefore $\triangle ABC$ is similar to $\triangle A'B'C'$ (306).

Second. In $\triangle ABC$ and $A'B'C'$, $\frac{BC}{B'C'} = \frac{AC}{A'C'}$ (313).

In the similar polygons, $\frac{BC}{B'C'} = \frac{CD}{C'D'}$ (313).

Consequently $\frac{AC}{A'C'} = \frac{CD}{C'D'}$ (Ax. 1).

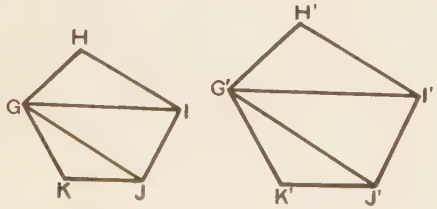
In the polygons, $\angle BCD = \angle B'C'D'$
In the $\triangle ABC$ and $A'B'C'$, $\angle BCA = \angle B'C'A'$ } (312).
Hence, by subtraction, $\angle ACD = \angle A'C'D'$ (Ax. 2).
Therefore $\triangle ACD$ is similar to $\triangle A'C'D'$ (306).

Third. $\triangle AED$ is proved similar to $\triangle A'E'D'$ in like manner. Q.E.D.

Ex. A map is drawn on a scale of 1 in. to 400 miles. How far is Paris from Antwerp if they are $1\frac{1}{8}$ in. apart on the map?

Proposition XXXI. Theorem

319. **If two polygons are composed of triangles similar each to each and similarly placed, the polygons are similar.** [Converse.]



Given: $\triangle GHI$ similar to $\triangle G'H'I'$;
$\triangle GIJ$ similar to $\triangle G'I'J'$;
$\triangle GJK$ similar to $\triangle G'J'K'$.

To Prove: The polygons HK and $H'K'$ similar.

Proof: First. In $\triangle HGI$ and $H'G'I'$, $\angle H = \angle H'$ (312).
Also in these $\triangle$ $\angle HIG = \angle H'I'G'$ (312).
In $\triangle GIJ$ and $G'I'J'$, $\angle GIJ = \angle G'I'J'$ (312).
Adding, $\angle HIJ = \angle H'I'J'$ (Ax. 2).
Likewise $\angle IJK = \angle I'J'K'$; etc.
That is, the polygons are mutually equiangular.

Second. In $\triangle GHI$ and $G'H'I'$, $\frac{GH}{G'H'} = \frac{HI}{H'I'} = \frac{GI}{G'I'}$ (313).

In $\triangle GIJ$ and $G'I'J'$, $\frac{GI}{G'I'} = \frac{IJ}{I'J'} = \frac{GJ}{G'J'}$ (?).

In $\triangle GJK$ and $G'J'K'$, $\frac{GJ}{G'J'} = \frac{JK}{J'K'} = \frac{KG}{K'G'}$ (?).

$$\therefore \frac{GH}{G'H'} = \frac{HI}{H'I'} = \frac{IJ}{I'J'} = \frac{JK}{J'K'} = \frac{KG}{K'G'} \quad \text{(Ax. 1).}$$

That is, the homologous sides are proportional.

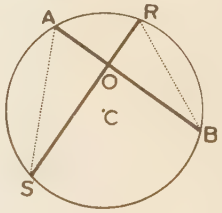
$\therefore$ The polygons are similar (301).

Q.E.D.

Ex. If two parallelograms are similar, and a diagonal in each is drawn, are the resulting triangles necessarily similar in pairs?

Proposition XXXII. Theorem

320. If through a fixed point within a circle two chords are drawn, the product of the segments of one equals the product of the segments of the other.



Given: Point O in circle C; chords AB and RS intersecting at O. (Review the note, p. 149.)

To Prove: $AO \cdot OB = RO \cdot OS$.

Proof: Draw AS and RB.

In $\triangle AOS$ and ROB, $\angle S = \angle B$ (239).

And $\angle A = \angle R$ (?).

$\therefore$ these $\triangle$ are similar (303).

$\therefore AO : RO = OS : OB$ (313).

$\therefore AO \cdot OB = RO \cdot OS$ (280). Q.E.D.

321. Corollary. The product of the segments of any chord drawn through a fixed point within a circle is constant for all chords through this point.

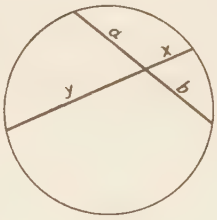
322. Direct proportion and reciprocal (or inverse) proportion.

Illustrations. I. If a man earns \$4½ each day, in 8 days he earns \$36. In 12 days he earns \$54. Hence, 8 da. : 12 da. = \$36 : \$54 is a proportion in which the antecedents belong to the same condition or circumstance, and the consequents belong to some other condition or circumstance. This is called a *direct proportion.*

II. If one man can build a certain wall in 120 days, 8 men can build it in 15 days; or 12 men in 10 days. Hence, 8 men : 12 men = 10 da. : 15 da. is a proportion in which the means belong to the same condition or circumstance, and the extremes belong to some other condition or circumstance. This is called a *reciprocal* (or *inverse*) *proportion.*

Definitions. A **direct proportion** is a proportion in which the antecedents belong to the same circumstance or figure, and the consequents belong to some other circumstance or figure. Thus, the ordinary proportions derived from similar figures are direct proportions.

A reciprocal (or **inverse**) **proportion** is a proportion in which the means belong to the same circumstance or figure, and the extremes belong to some other circumstance or figure.



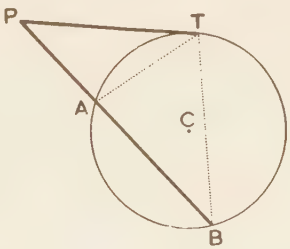
Thus, in the adjoining figure, $a \cdot b = x \cdot y$ (320). $\therefore a : x = y : b$ (281). This is a reciprocal proportion because the means are parts of *one* chord, and the extremes are parts of the *other* chord.

323. THEOREM. **If through a fixed point within a circle two chords are drawn, their four segments are reciprocally** (or **inversely**) **proportional.**

Proof: [Identical with 320; omitting the last step.]

PROPOSITION XXXIII. THEOREM

324. If from a fixed point without a circle a secant and a tangent are drawn, the product of the whole secant and the external segment equals the square of the tangent.



Given: $\odot C$; secant PAB; tangent PT.

To Prove: $PB \cdot PA = \overline{PT}^2$.

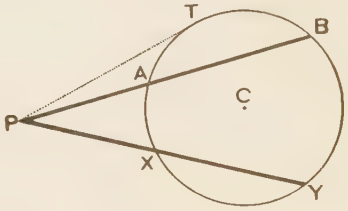
Proof: Draw AT and BT.

In $\triangle PAT$ and PBT,	$\angle P = \angle P$	(Iden.).
	$\angle PTA$ is measured by $\frac{1}{2}$ arc AT	(241).
	$\angle B$ is measured by $\frac{1}{2}$ arc AT	(236).
	$\therefore \angle PTA = \angle B$	(237).
Therefore	$\triangle PAT$ is similar to $\triangle PBT$	(303).
Hence	$PB : PT = PT : PA$	(313).
	$\therefore PB \cdot PA = \overline{PT}^2$	(280).

Q.E.D.

325. Corollary. **If from a fixed point without a circle any secant is drawn, the product of the secant and its external segment is constant for all secants.**

Proof: Any secant × ext. seg. = $(\text{tan.})^2$ = constant.



326. Corollary. **If from a fixed point without a circle two secants are drawn, these secants and their external segments are reciprocally (or inversely) proportional.**

Proof:

$$PB \cdot PA = PY \cdot PX \quad (325).$$

$$\therefore PB : PY = PX : PA \quad (281).$$

Q.E.D.

327. Theorem. **If from a fixed point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the secant and its external segment.**

Proof: [Identical with proof of 324; omitting the last step.]

Ex. 1. If $PA = 3$ in., and $PB = 12$ in., find the length of PT.

Ex. 2. If $PB = 21$ in., $PY = 15$ in., and $PA = 5$ in., find PX.

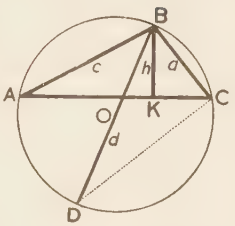
Ex. 3. Two altitudes of a triangle are reciprocally proportional to the bases to which they are drawn.

Ex. 4. If AD and BE are two altitudes of a triangle, and DE is drawn, the triangle ABC is similar to the triangle CED. [Use 306.]

Ex. 5. The four segments of the diagonals of a trapezoid are proportional.

Proposition XXXIV. Theorem

328. **In any triangle the product of two sides is equal to the diameter of the circumscribed circle multiplied by the altitude upon the third side.**



Given: $\triangle ABC$; circumscribed $\odot O$; altitude BK.

To Prove: $a \cdot c = d \cdot h$.

Proof: Draw chord CD. $\angle BCD =$ rt. $\angle$ (240).

In rt. $\triangle ABK$ and BCD, $\angle A = \angle D$ (239).

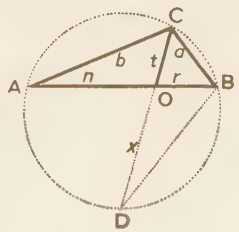
$\therefore$ these $\triangle$ are similar (304).

$\therefore c : d = h : a$ (313).

$\therefore a \cdot c = d \cdot h$ (280). Q.E.D.

Proposition XXXV. Theorem

329. **In any triangle the product of two sides is equal to the square of the bisector of their included angle, plus the product of the segments of the third side formed by the bisector.**



Given: $\triangle ABC$, CO bisector of $\angle ACB$.

To Prove: $a \cdot b = t^2 + n \cdot r$.

Proof: Circumscribe a $\odot$ about the $\triangle ABC$.

Produce CO to meet $\odot$ at D. Draw BD.

In $\triangle ACO$ and BCD, $\angle ACO = \angle BCD$ (Hyp.).

Also $\angle A = \angle D$ (239).

$\therefore \triangle ACO$ and BCD are similar (303).

$\therefore b : (t + x) = t : a$ (313).

$\therefore a \cdot b = t^2 + tx$ (280).

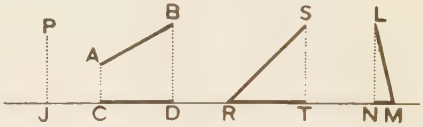
Now AB and CD are chords (Const.).

$\therefore t \cdot x = n \cdot r$ (320).

Substituting, $a \cdot b = t^2 + n \cdot r$ (Ax. 6). Q.E.D.

330. The **projection** of a **point** upon a line is the foot of the perpendicular from the point to the line.

Thus, the projection of P is J.



The **projection** of a definite **line** upon an indefinite line is the part of the indefinite line between the feet of the two perpendiculars to it, from the extremities of the definite line.

The projection of AB is CD; of RS is RT; of LM is NM.

Proposition XXXVI. Theorem

331. If in a right triangle a perpendicular is drawn from the vertex of the right angle upon the hypotenuse:

I. The triangles formed are similar to the given triangle and similar to each other.

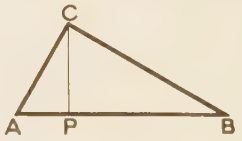
II. The perpendicular is a mean proportional between the segments of the hypotenuse.

Given: Rt. $\triangle ABC$; $CP \perp$ to AB from C.

To Prove:

I. $\triangle APC$, ABC, and BPC similar.

II. $AP : CP = CP : PB$.



Proof: I. In rt. $\triangle APC$ and ABC, $\angle A = \angle A$ (Iden.).

$\therefore$ $\triangle APC$ is similar to $\triangle ABC$ (304).

In rt. $\triangle BPC$ and ABC, $\angle B = \angle B$ (Iden.).

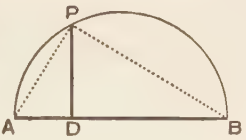
$\therefore$ $\triangle BPC$ is similar to $\triangle ABC$ (304).

Therefore $\triangle APC$, ABC, and BPC are all similar (307).

II. In the $\triangle APC$ and BPC, $AP : CP = CP : PB$ (313).

Q.E.D.

332. Corollary. If from any point in a circumference a perpendicular is drawn to a diameter, it is a mean proportional between the segments of the diameter.



Given: (?). **To Prove:** (?).

Proof: Draw chords AP and BP.

$\triangle APB$ is a rt. $\triangle$ (240).

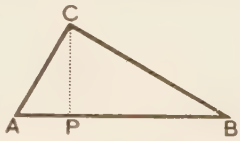
$\therefore AD : PD = PD : DB$ (331, II).

Q.E.D.

Proposition XXXVII. Theorem

333. The square of a leg of a right triangle is equal to the product of the hypotenuse and the projection of this leg upon the hypotenuse.

Given: Rt. $\triangle ABC$; AC and BC the legs.



To Prove: I. $\overline{AC}^2 = AB \cdot AP$.

II. $\overline{BC}^2 = AB \cdot BP$.

Proof: I. The rt. $\triangle ABC$ and APC are similar (331, I).

$\therefore AB : AC = AC : AP$ (313).

$\therefore \overline{AC}^2 = AB \cdot AP$ (280).

II. The rt. $\triangle ABC$ and BCP are similar (?).

$\therefore AB : BC = BC : BP$ (313).

$\overline{BC}^2 = AB \cdot BP$ (280).

Q.E.D.

Ex. 1. If, in 331, $AP = 3$, $PB = 27$, find CP.

Ex. 2. If, in 333, $AP = 4$, $PB = 21$, find AC and BC.

Ex. 3. If, in 333, $AB = 20$, $AC = 10$, find AP, BP, CP, and BC.

Ex. 4. If, in 331, $AP = 10$ and $CP = 20$, find BP.

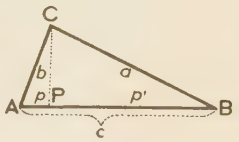
Ex. 5. If, in 333, $AB = 45$ and $AC = 15$, find AP, BP, CP, and BC.

Ex. 6. If, in 333, $AP = 9$ and $BP = 16$, find AC, PC, and BC.

Ex. 7. A stone arch in the shape of the arc of a circle is 4 ft. high. The chord of half the arch is 10 ft. Find the diameter.

Proposition XXXVIII. Theorem

334. The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse.



Given: Rt. $\triangle ABC$.

To Prove: $a^2 + b^2 = c^2$.

Proof: Draw $CP \perp$ to the hypotenuse AB.

Denote AP by p and PB by p'.

Now	$b^2 = c \cdot p$	(333).
Also	$a^2 = c \cdot p'$	(?).
Adding,	$a^2 + b^2 = c \cdot p + c \cdot p'$	(Ax. 2).
	$= c(p + p')$	
	$= c \cdot c = c^2$.	Q.E.D.

335. Corollary. The square of either leg of a right triangle is equal to the square of the hypotenuse minus the square of the other leg.

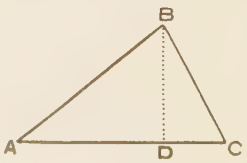
That is, $a^2 = c^2 - b^2$; and $b^2 = c^2 - a^2$ (Ax. 2).

Ex. 1. If $AC = 28$ and $BC = 45$, find AB.

Ex. 2. If $AC = 21$ and $AB = 29$, find BC.

Ex. 3. The square of the altitude of an equilateral triangle equals three fourths the square of a side.

Ex. 4. In any triangle the difference of the squares of two sides is equal to the difference of the squares of their projections on the third side.

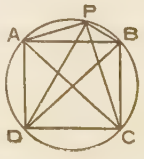


[$\overline{AB}^2 = (?)$; $\overline{BC}^2 = (?)$. Subtract, etc.]

Ex. 5. If the altitudes of triangle ABC meet at O, $\overline{AB}^2 - \overline{AC}^2 = \overline{BO}^2 - \overline{CO}^2$.

[Consult Ex. 4 and substitute.]

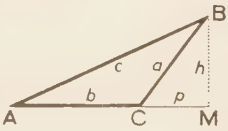
Ex. 6. If lines are drawn from any point in a circle to the four vertices of an inscribed square, the sum of the squares of these four lines is equal to twice the square of the diameter.



Proof: $\triangle APC$, DPB are rt. $\triangle$; etc.

Proposition XXXIX. Theorem

336. In an obtuse triangle the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides plus twice the product of one of these two sides and the projection of the other side upon that one.



Given: Obtuse $\triangle ABC$; etc.

To Prove: $c^2 = a^2 + b^2 + 2bp$.

Proof: In right $\triangle CBM$, $h^2 + p^2 = a^2$ (334).

In right $\triangle ABM$, $c^2 = h^2 + (p + b)^2$ (334).

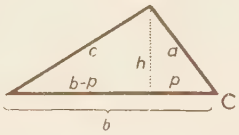
$$= \underbrace{h^2 + p^2} + b^2 + 2bp$$

$$= \quad a^2 \quad + b^2 + 2bp \quad \text{(Ax. 6).}$$

Q.E.D.

Proposition XL. Theorem

337. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides minus twice the product of one of these two sides and the projection of the other side upon that one.



Given: (?).

To Prove: $c^2 = a^2 + b^2 - 2bp$.

Proof: In one rt. $\triangle$, $h^2 + p^2 = a^2$ (334).

In the other rt. $\triangle$, $c^2 = h^2 + (b - p)^2$ (334).

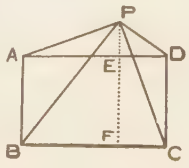
$$= \underbrace{h^2 + p^2} + b^2 - 2bp$$

$$= \quad a^2 \quad + b^2 - 2bp \quad \text{(Ax. 6).}$$

Q.E.D.

NOTE. This theorem is equally true in the case of an obtuse triangle.

Ex. If lines are drawn from any external point to the vertices of a rectangle $ABCD$, the sum of the squares of two of them which are drawn to a pair of opposite vertices is equal to the sum of the squares of the other two.



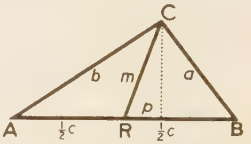
To Prove: $\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2$.

Proposition XLI. Theorem

338. I. The sum of the squares of two sides of a triangle is equal to twice the square of half the third side increased by twice the square of the median upon that side.

II. The difference of the squares of two sides of a triangle is equal to twice the product of the third side by the projection of the median upon that side.

Given: $\triangle ABC$; median $= m$; its projection $= p$; and side $b >$ side a.



To Prove:

I. $b^2 + a^2 = 2(\frac{1}{2}c)^2 + 2m^2$.

II. $b^2 - a^2 = 2cp$.

Proof: In $\triangle ARC$ and BRC, $AR = BR$ (Hyp.).

And $CR = CR$ (Iden.).

Also $AC > BC$ (Hyp.).

$\therefore \angle ARC > \angle BRC$ (92).

That is, $\angle ARC$ is obtuse and $\angle BRC$ is acute.

$\therefore$ In $\triangle ARC$, $b^2 = (\frac{1}{2}c)^2 + m^2 + cp$ (336).

In $\triangle BRC$, $a^2 = (\frac{1}{2}c)^2 + m^2 - cp$ (337).

I. Adding, $b^2 + a^2 = 2(\frac{1}{2}c)^2 + 2m^2$ (Ax. 2).

II. Subtracting, $b^2 - a^2 = 2cp$ (Ax. 2).

Q.E.D.

339. Formulas. If the vertices of a triangle are denoted by A, B, C, the lengths of the sides opposite are denoted by a, b, c, respectively; the altitude upon these sides by h_a, h_b h_c, respectively; the bisectors of the angles by t_a, t_b, t_c, respectively; the medians by m_a, m_b, m_c, respectively; the segments of the sides formed by the bisectors of the opposite angles by n_a and r_a, n_b and r_b, n_c and r_c; and the projections as follows: the projection of side a upon side b, by ${}_ap_b$; of side a upon side c, by ${}_ap_c$; of side b upon side c, by ${}_bp_c$; etc.

It is assumed that a, b, c are known. The following values of the various lines in a triangle are obtained by solving the equations already established.

I. **Projections.**

1. If $\angle C$ is obtuse, ${}_ap_b = \dfrac{c^2 - a^2 - b^2}{2b}$; ${}_bp_a = \dfrac{c^2 - a^2 - b^2}{2a}$; etc.

2. If $\angle C$ is acute, ${}_ap_b = \dfrac{a^2 + b^2 - c^2}{2b}$; ${}_bp_a = \dfrac{a^2 + b^2 - c^2}{2a}$; etc.

II. **Altitudes.** $h_b = \sqrt{a^2 - {}_ap_b{}^2}$; $h_a = \sqrt{b^2 - {}_bp_a{}^2}$; etc.

III. **Medians.** $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$; $m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}$; etc.

IV. **Bisectors.** $t_c = \sqrt{ab - n_c r_c}$*; $t_a = \sqrt{bc - n_a r_a}$*; $t_b = \sqrt{ac - n_b r_b}$.*

V. **Diameter** of circumscribed circle $= \dfrac{ac}{h_b}$; $= \dfrac{ab}{h_c}$; $= \dfrac{bc}{h_a}$.

VI. **Largest Angle.** Denote largest angle by C.

1. If $c^2 = a^2 + b^2$, $\angle C$ is right (334).
2. If $c^2 = a^2 + b^2$ *plus* something, $\angle C$ is obtuse (336).
3. If $c^2 = a^2 + b^2$ *minus* something, $\angle C$ is acute (337)

Ex. 1. If the sides of a triangle are $a = 7$, $b = 10$, $c = 12$, find the nature of $\angle C$.

Ex. 2. In the same triangle find m_a. Find m_b. Find m_c.

Ex. 3. In the same triangle find ${}_ap_b$. Find ${}_bp_a$. Find ${}_ap_c$. Find ${}_bp_c$. Find ${}_cp_a$. Find ${}_cp_b$.

Ex. 4. Find h_a. Find h_b. Find h_c.

Ex. 5. Find the diameter of the circumscribed circle.

Ex. 6. Find n_a and r_a. Find n_b and r_b. Find n_c and r_c.

Ex. 7. Find t_a. Find t_b. Find t_c.

CONCERNING ORIGINALS

340. First determine from the nature of each numerical exercise upon which theorem it depends, and then apply that theorem.

* The segments n and r can be found by 297 ; $n_b : r_b = c : a$, etc.

ORIGINAL EXERCISES (NUMERICAL)

1. The legs of a right triangle are 12 and 16 inches. Find the hypotenuse.

2. The side of a square is 6 feet. What is the diagonal?

3. The base of an isosceles triangle is 16 and the altitude is 15. Find the equal sides.

4. The tangent to a circle from a point is 12 inches and the radius of the circle is 5 inches. Find the length of the line joining the point to the center.

5. In a circle whose radius is 13 inches, what is the length of a chord 5 inches from the center?

6. The length of a chord is 2 feet and its distance from the center is 35 inches. Find the radius of the circle.

7. The hypotenuse of a right triangle is 2 feet 2 inches, and one leg is 10 inches. Find the other.

8. The base of an isosceles triangle is 90 inches and the equal sides are each 53 inches. Find the altitude.

9. The radius of a circle is 4 feet 7 inches. Find the length of the tangent drawn from a point 6 feet 1 inch from the center.

10. How long is a chord 21 yards from the center of a circle whose radius is 35 yards?

11. Each side of an equilateral triangle is 4 feet. Find the altitude.

12. The altitude of an equilateral triangle is 8 feet. Find the side.

13. Each side of an isosceles right triangle is a. Find the hypotenuse.

14. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centers?

15. The diagonal of a rectangle is 82 and one side is 80. Find the other.

16. The length of a tangent to a circle whose diameter is 20, from an external point, is 24. What is the distance from this point to the center?

17. The diagonal of a square is 10. Find each side.

18. Find the length of a chord 2 feet from the center of a circle whose diameter is 5 feet.

19. A flagpole was broken 16 feet from the ground, and the top struck the ground 63 feet from the foot of the pole. How long was the pole?

20. The top of a ladder 17 feet long reaches a point on a wall 15 feet from the ground. How far is the lower end of the ladder from the wall?

21. A chord 2 feet long is 5 inches from the center of a circle. How far from the center is a chord 10 inches long? [Find the radius.]

22. The diameters of two concentric circles are 1 foot 10 inches and 10 feet 2 inches. Find the length of a chord of the larger which is tangent to the less.

23. The lower ends of a post and a flagpole are 42 feet apart; the post is 8 feet high and the pole, 48 feet. How far is it from the top of one to the top of the other?

24. The radii of two circles are 8 inches and 17 inches, and their centers are 41 inches apart. Find the lengths of their common external tangents; of their common internal tangents.

25. A ladder 65 feet long stands in a street. If it inclines toward one side, it will touch a house at a point 16 feet above the pavement; if to the other side, it will touch a house at a point 56 feet above the pavement. How wide is the street?

26. Two parallel chords of a circle on opposite sides of the center are 4 feet, and 40 inches long, respectively, and the distance between them is 22 inches. Find the radius of the circle.

[Draw the radii to ends of chords; these = hypotenuses = R; the distances from the center = x and $22 - x$.]

27. The legs of an isosceles trapezoid are each 2 ft. 1 in. long, and one of the bases is 3 ft. 4 in. longer than the other. Find the altitude.

28. One of the non-parallel sides of a trapezoid is perpendicular to both bases, and is 63 feet long; the bases are 41 feet and 25 feet long. Find the length of the remaining side.

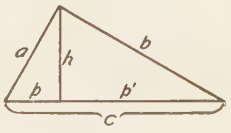
29. If $a = 10$, $h = 6$, find p, c, p', b.

30. If $h = 8$, $p' = 4$, find b, c, p, a.

31. If $a = 10$, $p' = 15$, find p, c, h, b.

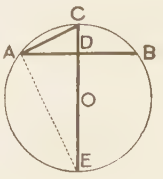
32. If $a = 9$, $b = 12$, find c, p, p', h.

33. If $p = 3$, $p' = 12$, find h, a, b.



34. The line joining the midpoint of a chord to the midpoint of its arc is 5 inches. If the chord is 2 feet long, what is the diameter?

[$\triangle ACE$ is rt. $\triangle$ (?). $\therefore \overline{AC}^2 = CE \cdot CD$ (?).]



35. If the chord of an arc is 60 and the chord of its half is 34, what is the diameter?

36. The line joining the midpoint of a chord to the midpoint of its arc is 6 inches. The chord of half this arc is 18 inches. Find the diameter. Find the length of the original chord.

37. To a circle whose radius is 10 inches, two tangents each 2 feet long are drawn from a point. Find the length of the chord joining their points of contact.

38. The sides of a triangle are 6, 9, 11. Find the segments of the shortest side made by the bisector of the opposite angle.

39. Find the segments of the longest side made by the bisector of the largest angle in Ex. 38.

40. The sides of a triangle are 5, 9, 12. Find the segments of the shortest side made by the bisector of the opposite exterior angle; also of the medium side made by the bisector of its opposite exterior angle.

41. In the figure of 295, if $AC = 3$, $CE = 5$, $EG = 8$, $BD = 4$, find DF and FH.

42. If the sides of a triangle are 6, 8, 12 and the shortest side of a similar triangle is 15, find its other sides.

43. If the homologous altitudes of two similar triangles are 9 and 15 and the base of the former is 21, what is the base of the latter?

44. In the figure of 315, $AE = 4$, $EF = 6$, $FB = 9$, $GH = 15$. Find CG and CD.

45. The sides of a pentagon are 5, 6, 8, 9, 18, and the longest side of a similar pentagon is 78. Find the other sides.

46. A pair of homologous sides of two similar polygons are 9 and 16. If the perimeter of the first is 117, what is the perimeter of the second?

47. The perimeters of two similar polygons are 72 and 120. The shortest side of the former is 4. What is the shortest side of the latter?

48. Two similar triangles have homologous bases 20 and 48. If the altitude of the latter is 36, find the altitude of the former.

49. The segments of a chord, made by a second chord, are 4 and 27. One segment of the second chord is 6. Find the other.

50. One of two intersecting chords is 19 inches long and the segments of the other are 5 inches and 12 inches. Find the segments of the first chord.

51. Two secants are drawn to a circle from a point; their lengths are 15 inches and $10\frac{1}{2}$ inches. The external segment of the latter is 10 inches. Find the external segment of the former.

52. The tangent to a circle is 1 foot long and the secant from the same point is 1 foot 6 inches. Find the chord part of the secant.

53. The internal segment of a secant 25 inches long is 16 inches. Find the tangent from the same point to the same circle.

54. Two secants to a circle from a point are $1\frac{1}{2}$ feet and 2 feet long; the tangent from the same point is 12 inches. Find the external segments of the two secants.

55. If the sides of a triangle are 5, 6, 8, is the angle opposite 8 right, acute, or obtuse? if the sides are 8, 7, 4?

56. If the sides of a triangle are 8, 9, 12, is the largest angle right, acute, or obtuse? if the sides are 13, 7, 11?

57. The sides of a triangle are x, y, z. If z is the greatest side, when will the angle opposite be right? obtuse? acute?

58. The sides of a triangle are 6, 8, 9. Find the length of the projection of side 6 upon side 8; of side 8 upon side 9; of side 9 upon side 6.

59. The sides of a triangle are 5, 6, 9. Find the length of the projection of side 6 upon side 5; of side 9 upon side 6.

60. Find the three altitudes in a triangle with sides 9, 10, 17.

61. Find the three altitudes in a triangle with sides 11, 13, 20.

62. Find the diameter of a circle circumscribed about a triangle with sides 17, 25, 26.

63. Find the length of the bisector of the least angle of a triangle with sides 7, 15, 20; also of the largest angle.

64. Find the length of the bisector of the largest angle of a triangle with sides 12, 32, 33; also of the other angles.

65. Find the three medians in a triangle with sides 4, 7, 9.

66. Find the product of the segments of every chord drawn through a point 4 units from the center of a circle whose radius is 10 units.

67. The bases of a trapezoid are 12 and 20, and the altitude is 8. The other sides are produced to meet. Find the altitude of the larger triangle formed.

68. The shadow of a yardstick perpendicular to the ground is $4\frac{1}{2}$ feet. Find the height of a tree whose shadow at the same time is 100 yards.

69. There are two belt-wheels 3 feet 8 inches and 1 foot 2 inches in diameter, respectively. Their centers are 9 feet 5 inches apart. Find the length of the belt suspended between the wheels if the belt does not cross itself; also the length of the belt if it does cross.

SUMMARY

341. Triangles are proved **similar** by showing that they have:

(1) Two angles of one equal to two angles of the other.

(2) An acute angle of one equal to an acute angle of the other. [In right triangles.]

(3) Homologous sides proportional.

(4) An angle of one equal to an angle of the other and the including sides proportional.

(5) Their sides respectively parallel or perpendicular.

Four lines are proved **proportional** by showing that they are:

(1) Homologous sides of similar triangles.

(2) Homologous sides of similar polygons.

(3) Homologous lines of similar figures.

The product of two lines is proved equal to the product of two others, by proving these four lines proportional and making the product of the extremes equal to the product of the means.

One line is proved a **mean proportional** between two others by proving that two triangles containing this line in common are similar, and obtaining the proportion from their sides.

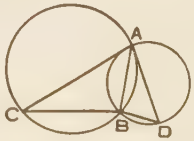
In cases dealing with the **square of a line**, one uses:

(1) Similar triangles having this line in common, or,

(2) A right triangle containing this line as a part.

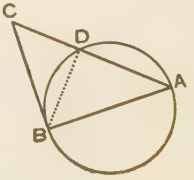
ORIGINAL EXERCISES (THEOREMS)

1. In any right triangle the product of the hypotenuse and the altitude upon it is equal to the product of the legs.

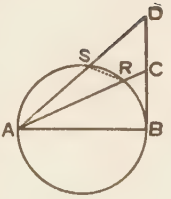


2. If two circles intersect at A and B, and AC and AD are drawn, each a tangent to one circle and a chord of the other, the common chord AB is a mean proportional between BC and BD.

3. If AB is a diameter and BC a tangent, and AC meets the circumference at D, the diameter is a mean proportional between AC and AD.



4. If two circles are tangent externally, the chords formed by a straight line drawn through their point of contact have the same ratio as the diameters of the circles.



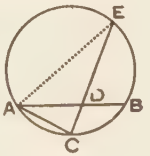
5. If a tangent is drawn from one extremity of a diameter, meeting secants from the other extremity, these secants and their internal segments are reciprocally proportional.

To Prove: $AC : AD = AS : AR$.

Proof: Draw RS. In $\triangle ARS$ and ACD, $\angle A = \angle A$ and $\angle ARS = \angle D$. (Explain.) Etc.

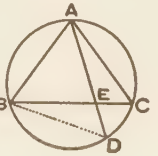
6. If AB is a chord and CE, another chord, drawn from C, the midpoint of arc AB, meeting chord AB at D, AC is a mean proportional between CD and CE.

Prove the above theorem and deduce that, $CE \cdot CD$ is constant for all positions of the point E on arc AEB

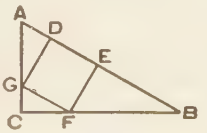


7. If chord AD is drawn from vertex A of inscribed isosceles triangle ABC, cutting BC at E, AB is a mean proportional between AD and AE.

Prove the above theorem and deduce that, $AD \cdot AE$ is constant for all positions of the point D on arc BDC.



8. If a square is inscribed in a right triangle so that one vertex is on each leg of the triangle and the other two vertices on the hypotenuse, the side of the square is a mean proportional between the other segments of the hypotenuse.



To Prove: $AD : DE = DE : EB$.

First prove $\triangle ADG$ and BEF similar.

9. If the sides of two unequal triangles are respectively parallel, the lines joining homologous vertices meet in a point. (These lines to be produced if necessary.)

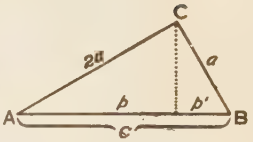
10. AB is any chord; AC is a tangent and CDE is a secant parallel to AB cutting the circle at D and E. Prove that $AC : AE = DC : BE$.

11. Prove theorem of 320, by drawing two other auxiliary lines.

12. Prove theorem of 316 if point O is between the parallels.

13. Prove theorem of 326 by drawing auxiliary lines AY and BX.

14. If one leg of a right triangle is double the other, its projection upon the hypotenuse is four times the projection of the other.



Proof: $(2a)^2 = cp$; $a^2 = cp'$ (?).

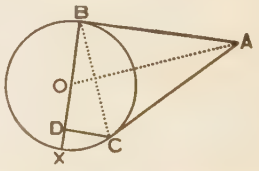
$\therefore p = \frac{4a^2}{c}$; $p' = \frac{a^2}{c}$ (Ax. 3). $\therefore p = 4p'$ (?).

15. If the bisector of an angle of a triangle bisects the opposite side, the triangle is isosceles.

16. The tangents to two intersecting circles from any point in their common chord produced are equal. [Use 324.]

17. If two circles intersect, their common chord, produced, bisects their common tangents.

18. If AB and AC are tangents to a circle from A; CD is perpendicular to diameter BOX from C; then $AB \cdot CD = BD \cdot BO$.



19. If the altitude of an equilateral triangle is h, find the side.

20. If one side of a triangle is divided by a point into segments which are proportional to the other sides, a line from this point to the opposite angle bisects that angle.

To Prove: $\angle n = \angle m$ in figure of 297.

Proof: Produce CB to P, making $BP = AB$; draw AP; etc.

21. State and prove the converse of 298.

22. Two rhombuses are similar if an angle of one is equal to an angle of the other.

23. If two circles are tangent internally and any two chords of the greater are drawn from their point of contact, they are divided proportionally by the less circle.

[Draw diameter to point of contact and prove the right $\triangle$ similar.]

24. The non-parallel sides of a trapezoid and the line joining the midpoints of the bases, if produced, meet at a point. [Use Ax. 3 and 316.]

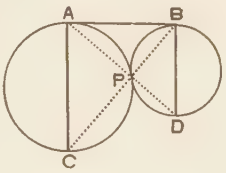
25. The diagonals of a trapezoid and the line joining the midpoint of the bases meet at a point.

26. If one chord bisects another, either segment of the latter is a mean proportional between the segments of the other.

27. Two parallelograms are similar if they have an angle of the one equal to an angle of the other and the including sides proportional.

28. Two rectangles are similar if two adjoining pairs of homologous sides are proportional.

29. If two circles are tangent externally, the common exterior tangent is a mean proportional between the diameters.



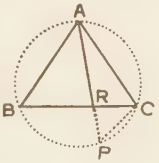
[Draw chords PA, PC, PB, PD.
Prove $\angle APB$ a rt. $\angle$.
Then prove APD and BPC straight lines.
Then prove $\triangle ABC$ and ABD similar.]

30. In any rhombus the sum of the squares of the diagonals is equal to the square of half the perimeter.

31. If in an angle a series of parallel lines are drawn having their ends in the sides of the angle, their midpoints lie in one straight line.

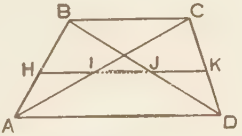
32. If ABC is an isosceles triangle and BX is the altitude upon AC (one of the legs), $\overline{BC}^2 = 2\,AC \cdot CX$. [Use 337.]

33. In an isosceles triangle the square of one leg is equal to the square of the line drawn from the vertex to any point of the base, plus the product of the segments of the base.



Proof: Circumscribe a $\odot$; use method of 329.

34. If a line is drawn in a trapezoid parallel to the bases, the segments between the diagonals and the non-parallel sides are equal.



Proof: $\triangle AHI$ and ABC are similar (?); $\triangle DKJ$ and DCB also.

$\therefore \dfrac{AH}{AB} = \dfrac{HI}{BC}$, and $\dfrac{DK}{DC} = \dfrac{JK}{BC}$. (Explain.)

But $\dfrac{AH}{AB} = \dfrac{DK}{DC}$ (295). $\therefore \dfrac{HI}{BC} = \dfrac{JK}{BC}$ (Axiom 1). Etc.

35. A line through the point of intersection of the diagonals of a trapezoid, and parallel to the bases, is bisected by that point.

36. If M is the midpoint of hypotenuse AB of right triangle ABC, $\overline{AB}^2 + \overline{BC}^2 + \overline{AC}^2 = 8\,\overline{CM}^2$.

37. The squares of the legs of a right triangle have the same ratio as their projections upon the hypotenuse.

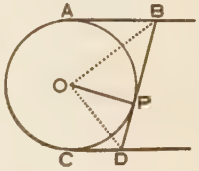
38. If the diagonals of a quadrilateral are perpendicular to each other, the sum of the squares of one pair of opposite sides is equal to the sum of the squares of the other pair.

39. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals. [Use 338, I.]

40. If DE is drawn parallel to the hypotenuse AB of right triangle ABC, meeting AC at D and CB at E, $\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2$.

[Use 4 rt. ∡ having vertex C.]

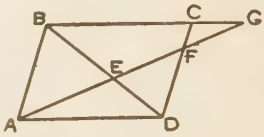
41. If between two parallel tangents a third tangent is drawn, the radius of the circle is a mean proportional between the segments of the third tangent.



To Prove: $BP : OP = OP : PD$.

Proof: $\triangle BOD$ is a rt $\triangle$ (?). Etc.

42. If $ABCD$ is a parallelogram, BD a diagonal, AG any line from A meeting BD at E, CD at F, and BC (produced) at G, AE is a mean proportional between EF and EG.



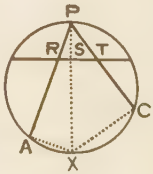
Proof: ∡ ABE and EDF are similar (?); also ∡ ADE and BEG (?). Obtain two ratios equal to $BE : ED$ and then apply Ax. 1.

43. An interior common tangent of two circles divides the line joining their centers into segments proportional to the radii.

44. An exterior common tangent of two circles divides the line joining their centers (externally) into segments proportional to the radii.

45. The common internal tangents of two circles and the common external tangents meet on the line determined by the centers of the circles.

46. If from the midpoint P, of an arc subtended by a given chord, chords are drawn cutting the given chord, the product of each whole chord from P and its segment adjacent to P is constant.

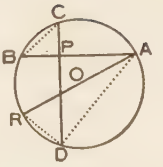


Proof: Take two such chords, PA and PC; draw diameter PX; etc. Rt. ∡ PST and PCX are similar. (Explain.)

47. If from any point within a triangle ABC, perpendiculars to the sides are drawn — OR to AB, OS to BC, OT to AC,

$\overline{AR}^2 + \overline{BS}^2 + \overline{CT}^2 = \overline{BR}^2 + \overline{CS}^2 + \overline{AT}^2$. [Draw OA, OB, OC.]

48. If two chords intersect within a circle and at right angles, the sum of the squares of their four segments equals the square of the diameter.



To Prove: $\overline{AP}^2 + \overline{BP}^2 + \overline{CP}^2 + \overline{DP}^2 = \overline{AR}^2$.

Proof: Draw BC, AD, RD. Chord BR is $\perp$ to AB (240).

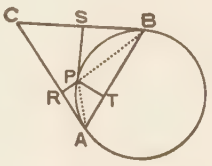
$\therefore$ CD is $\parallel$ to BR (?) $\quad\therefore$ arc BC = arc RD (?).

$\therefore$ chord BC = chord RD (?). Also $\triangle ARD$ is rt. $\triangle$ (?).

Now $\overline{AR}^2 = \overline{AD}^2 + \overline{DR}^2$ (?).

But $\overline{AD}^2 = \overline{AP}^2 + \overline{PD}^2$ and $\overline{DR}^2$ = etc.

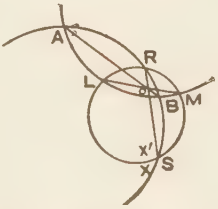
49. The perpendicular from any point of an arc upon its chord is a mean proportional between the perpendiculars from the same point to the tangents at the ends of the chord.



To Prove: $PR : PT = PT : PS$.

Proof: Prove $\triangle ARP$ and BTP similar; also $\triangle APT$ and PBS (?). Thus, get two ratios each = to $PA : PB$.

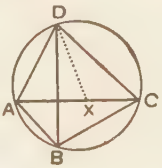
50. If each of three circles intersects the other two, the three common chords meet in a point.



To Prove: AB, LM, RS meet at O.

Proof: Suppose AB and LM meet at O. Draw RO and produce it to meet the ⓢ at X and X'. Prove $OX = OX'$ (by 320). $\therefore$ X, X', S are coincident.

51. In an inscribed quadrilateral the sum of the products of the two pairs of opposite sides is equal to the product of the diagonals.



Proof: Draw DX making $\angle CDX = \angle ADB$; $\triangle ADB$ and CDX are sim. (?); also $\triangle BCD$ and ADX (?).

Hence $\quad AB \cdot DC = DB \cdot XC$ (?),

Also $\quad AD \cdot BC = DB \cdot AX$ (?).

Adding, etc.

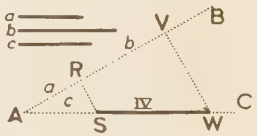
52. If AB is a diameter, BC and AD tangents, meeting chords AF and BF (produced) at C and D respectively, AB is a mean proportional between the tangents BC and AD.

53. If from a point A on the circumference of a circle two chords are drawn and a line parallel to the tangent at A meet them, the chords and their segments nearer to A are inversely proportional.

CONSTRUCTION PROBLEMS

Proposition XLII. Problem

342. **To find a fourth proportional to three given lines.**



Given: Three lines a, b, c.

Required: To find a fourth proportional to a, b, c.

Construction: Take two indefinite lines, AB and AC, meeting at A. On AB take $AR =$ to a, $RV =$ to b. On AC take $AS =$ to c. Draw RS.

From V draw $VW \parallel$ to RS, meeting AC at W.

Statement: SW is the fourth proportional required. Q.E.F.

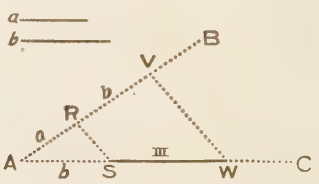
Proof: In $\triangle AVW$, RS is $\parallel$ to VW (Const.).

$\therefore a : b = c : SW$ (294).

Q.E.D.

Proposition XLIII. Problem

343. **To find a third proportional to two given lines.**



Given: (?).

Required: (?).

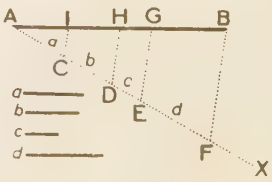
Construction:

Like that for 342.

Statement: (?). **Proof:** (?)

Proposition XLIV. Problem

344. **To divide a given line into segments proportional to any number of given lines.**



Given: AB; a, b, c, d.

Required: To divide AB into parts which shall be proportional to a, b, c, d.

Construction: Draw AX oblique to AB from A. On AX take $AC =$ to a, $CD =$ to b, $DE =$ to c, $EF =$ to d. Draw FB also through E, D, and C, lines $\parallel$ to FB, as EG, DH, and CI.

Statement: AI, IH, HG, GB are the required parts. Q.E.F.

Proof: $AI : a = IH : b = HG : c = GB : d$ (295).

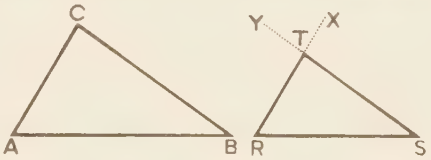
Q.E.D.

Proposition XLV. Problem

345. To construct a triangle similar to a given triangle and having a given side homologous to a side of the given triangle.

Given: $\triangle ABC$ and RS homologous to AB.

Required: To construct a $\triangle$ on RS similar to $\triangle ABC$.



Construction: At R construct $\angle SRX =$ to $\angle A$; at S construct $\angle RSY =$ to $\angle B$, the sides of these angles meeting at T.

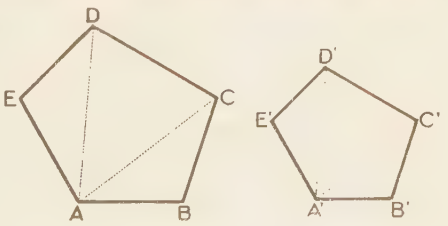
Statement: (?). **Proof:** (?). (303).

Proposition XLVI. Problem

346. To construct a polygon similar to a given polygon and having a given side homologous to a side of the given polygon.

Given: Polygon EB; line $A'B'$ homologous to AB.

Required: To construct a polygon upon $A'B'$, similar to polygon EB.



Construction: From A draw diagonals AC and AD.

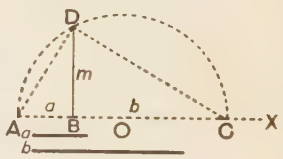
On $A'B'$ construct $\triangle A'B'C'$ similar to $\triangle ABC$ (by 345).

On $A'B'$ construct $\triangle A'C'D'$ similar to $\triangle ACD$. Etc.

Statement: (?). **Proof:** (?). (319).

Proposition XLVII. Problem

347. To find the mean proportional between two given lines.



Given: Lines a and b.

Required: To find the mean proportional between them.

Construction: On an indefinite line, AX, take $AB =$ to a and $BC =$ to b. Using O, the midpoint of AC, as center, and AO as radius, describe the semicircle, ADC. At B erect $BD \perp$ to AC, meeting the arc at D. Draw AD and CD.

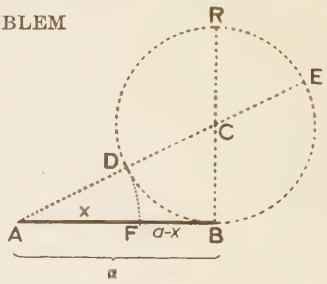
Statement: BD, or m, is the mean proportional required. Q.E.F.

Proof: $a : m = m : b$. (?) Q.E.D.

348. A line is divided into **extreme and mean ratio** if one segment is a mean proportional between the whole line and the other segment; in other words, if a line is to one of its parts as that part is to the other part. (See 292.)

Proposition XLVIII. Problem

349. To divide a line into extreme and mean ratio.



Given: Line $AB = a$.

Required: To divide AB into extreme and mean ratio; that is, so that $AB : AF = AF : FB$.

Construction: At B erect BR, $\perp$ to AB and $=$ to AB. Using C, the midpoint of BR, as center, and CB as radius, describe a $\odot$. Draw AC meeting $\odot$ at D and E. On AB take $AF = AD$; let $AF = x$.

Statement: F divides AB so that $AB : AF = AF : FB$. Q.E.F.

Proof: AB is tangent to $\odot C$ (202).

$\therefore AE \cdot AD = \overline{AB}^2$ (324).

Now $AD = x$ (187) and $DE = a$ (190).

$\therefore AE = a + x$ (Ax. 4).

Substituting, $(a+x)x = a^2$ (Ax. 6).

Or $ax + x^2 = a^2$

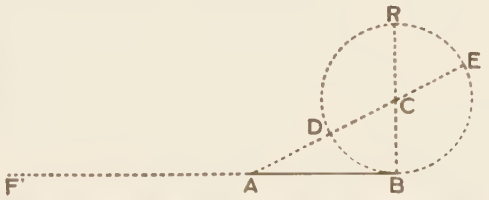
$\therefore x^2 = a^2 - ax = a(a-x)$

$\therefore a : x = x : a - x$ (281).

That is, $AB : AF = AF : FB$. Q.E.D.

Proposition XLIX. Problem.

350. To divide a line externally into extreme and mean ratio.



Given: (?).

Required: (?).

Construction: The same as in 349, except that AF' is taken on BA produced, $=$ to AE.

Statement: $AB : AF' = AF' : BF'$. Q. E. F.

Proof: AB is tangent to $\odot C$ (202).

$\therefore AE : AB = AB : AD$ (325).

$\therefore AE + AB : AE = AB + AD : AB$ (284).

Now $AE + AB = BF'$ (Ax. 4).

Also $AB + AD = AE = AF'$ (Const.).

Substituting, $BF' : AF' = AF' : AB$ (Ax. 6).

That is, $AB : AF' = AF' : BF'$. Q.E.D.

351. The lengths of the several lines of 349 and 350 may be found by algebra, if the length of AB is known.

Thus if $AB = a$, we know in 349, $a : x = x : a - x$.

Hence $x^2 = a^2 - ax$. Solving this quadratic,

$x = AF = \frac{1}{2}\, a(\sqrt{5} - 1)$; also, $a - x = BF = \frac{1}{2}\, a(3 - \sqrt{5})$.

Likewise in 350, if $AB = a$, $AF' = y$, $a : y = y : a + y$.

Solving for y, $y = AF' = \frac{1}{2}\, a(\sqrt{5} + 1)$.

Also $a + y = BF' = \frac{1}{2}\, a(3 + \sqrt{5})$.

ORIGINAL CONSTRUCTIONS

It is required:

1. To construct a fourth proportional to lines that are exactly 3 in., 5 in., and 6 in. long. How long should this constructed line be?

2. To construct a mean proportional between lines that are exactly 4 in. and 9 in. How long should this constructed line be?

3. To construct a fourth proportional to three lines 5 in., 8 in., and 10 in. will this be the same length as a fourth proportional to 5 in., 10 in., and 8 in.? to 8 in., 10 in., and 5 in.? to 10 in., 5 in., and 8 in.?

4. To construct a third proportional to lines 3 in. and 6 in. long.

5. To produce a given line AB to point P, such that $AB : AP = 3 : 5$. [Divide AB into three equal parts, etc.]

6. To divide a line 8 in. long into two parts in the ratio of 5 : 7. [Divide the given line into 12 equal parts.]

7. To solve Ex. 6 by constructing a triangle. [See 297.]

8. To divide one side of a triangle into segments proportional to the other two sides.

9. To divide one side of a triangle externally into segments proportional to the other sides.

10. To construct two straight lines having given their sum and ratio. [Consult Ex. 6.]

11. To construct two straight lines having given their difference and ratio. [Consult Ex. 5.]

12. To construct a triangle similar to a given triangle and having a given perimeter. [First, use 344.]

13. To construct a right triangle having given its perimeter and an acute angle. [Constr. a rt. $\triangle$ having the given acute $\angle$. Etc.]

14. To construct a triangle having given its perimeter and two angles. [Constr. a $\triangle$ having the two given $\measuredangle$. Etc.]

15. To construct a triangle similar to a given triangle and having a given altitude.

16. To construct a rectangle similar to a given rectangle and having a given base.

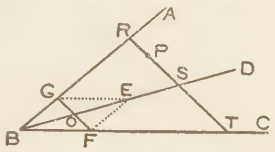
17. To construct a rectangle similar to a given rectangle and having a given perimeter.

18. To construct a parallelogram similar to a given parallelogram and having a given base.

19. To construct a parallelogram similar to a given parallelogram and having a given perimeter.

20. To construct a parallelogram similar to a given parallelogram, and having a given altitude.

21. To draw through a given point another line, which is terminated by the outer two of three lines meeting in a point and bisected by the inner one.



Construction: From E on BD draw ⊥s. Etc. Through P draw $RT \parallel$ to GF.

Statement: $RS = ST$.

22. To inscribe in a given circle a triangle similar to a given triangle.

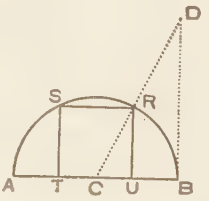
Construction: Circumscribe a ⊙ about the given △; draw radii to the vertices; at center of given ⊙ construct 3 ∠s = to the other central angles.

23. To circumscribe about a given circle a triangle similar to a given triangle.

Construction: First, inscribe a △ similar to the given △.

24. To construct a right triangle, having given one leg and its projection upon the hypotenuse.

25. To inscribe a square in a given semicircle.



Construction: At B erect $BD \perp$ to AB and = to AB; draw DC, meeting ⊙ at R; draw $RU \parallel$ to BD. Etc.

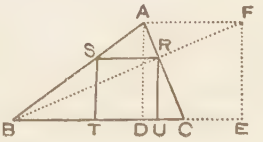
Statement: (?).

Proof: $RSTU$ is a rectangle (?). $\triangle CRU$ is similar to $\triangle CDB$ (?). $\therefore CU : CB = UR : BD$ (?). But $CB = \frac{1}{2} BD$ (?). $\therefore CU = \frac{1}{2} UR$ (?). Etc.

26. To inscribe in a given semicircle a rectangle similar to a given rectangle.

Construction: From the midpoint of the base draw line to one of the opposite vertices. At given center construct an ∠ = to the ∠ at the midpoint. Proceed as in Ex. 25.

27. To inscribe a square in a given triangle.



Construction: Draw altitude AD; construct the square $ADEF$ upon AD as a side; draw BF meeting AC at R.

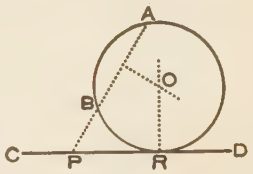
Draw $RU \parallel$ to AD; $RS \parallel$ to BC. Etc.

28. To inscribe in a given triangle a rectangle similar to a given rectangle.

Construction: Draw the altitude. On this construct a rectangle similar to the given rectangle.

Proceed as in Ex. 27.

29. To construct a circle which shall pass through two given points and touch a given line.



Given: Points A and B; line CD.

Construction: Draw line AB meeting CD at P. Construct a mean proportional between PA and PB (by 347). On PD take PR = to this mean. Erect $OR \perp$ to CD at R, meeting $\perp$ bisector of AB at O. Use O as center, etc.

30. To construct a line = to $\sqrt{2}$ in. [Diag. of square the side of which is 1 in.]

31. To construct a line = to $\sqrt{5}$ in.

[Hyp. of a rt. $\triangle$, whose legs are 1 in. and 2 in. respectively.]

32. To divide a line into segments in the ratio of $1 : \sqrt{2}$.

33. To divide a line into segments in the ratio of $1 : \sqrt{5}$.

34. To construct a line x, if $x = \frac{ab}{c}$, and a, b, c are lengths of three given lines. [That is, to construct x, if $c : a = b : x$ (281).]

35. To construct a line x, if $x = \frac{ab}{3c}$. [$3c : a = b : x$.]

36. To construct a line x, if $x = \sqrt{ab}$. [$a : x = x : b$.]

37. To construct a line x, if $x = \frac{a^2}{c}$.

38. To construct a line x, if $x = \sqrt{a^2 - b^2}$. [$a + b : x = x : a - b$.]

39. To construct a line x, if $x = \frac{2a^2}{c}$.

40. To construct a line y, if $ay = \frac{2}{3}b^2$.

41. To construct a line = to $\sqrt{10}$ in.

42. To construct a line = to $2\sqrt{6}$ in.

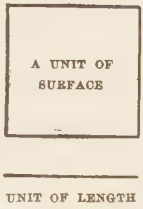
43. To construct a line = to $\sqrt{a^2 + b^2}$, if a and b are given lines.

BOOK IV

AREAS OF POLYGONS

THEOREMS AND DEMONSTRATIONS

352. The **unit of surface** is a square each side of which is a unit of length.



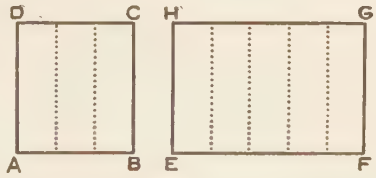
353. The **area of a surface** is the *number* of units of surface it contains. The area of a surface is the ratio of that surface to the unit of surface.

NOTE. It is often convenient to speak of "triangle," "rectangle," etc., when one really means "the area of a triangle," or "the area of a rectangle," etc.

PROPOSITION I. THEOREM

354. If two rectangles have equal altitudes, they are to each other as their bases.

Given: Rectangles AC and EG having equal altitudes, with bases AB and EF.



To Prove:

$AC : EG = AB : EF$.

Proof: I. If AB and EF are **commensurable**.

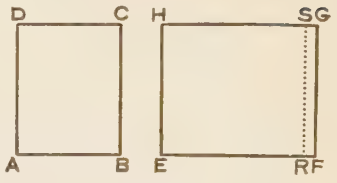
There is a common unit of measure of AB and EF (225). Suppose this unit is contained 3 times in AB and 5 times in EF. Hence $AB : EF = 3 : 5$ (Ax. 3).

Draw lines through these points of division $\perp$ to the bases. These divide rectangle AC into three parts and EG into 5 parts, and all of these eight parts are equal (134).

Hence $AC : EG = 3 : 5$ (Ax. 3).

$\therefore AC : EG = AB : EF$ (Ax. 1). Q.E.D.

II. If AB and EF are **incommensurable.**



There does not exist a common unit (225).

Divide AB into several equal parts. Apply one of these as a unit of measure to EF.

There is a remainder, RF (Hyp.).

Draw $RS \perp$ to EF. Now $\frac{AC}{ES} = \frac{AB}{ER}$ (Case I).

Indefinitely increase the number of equal parts of AB; that is, indefinitely decrease each part, or the unit or divisor. Then the remainder, RF, is indefinitely decreased.

That is, RF approaches zero as a limit,

Also $RFGS$ approaches zero as a limit.

Hence ER approaches EF as a limit,

Also ES approaches EG as a limit.

Therefore $\frac{AC}{ES}$ approaches $\frac{AC}{EG}$ as a limit,

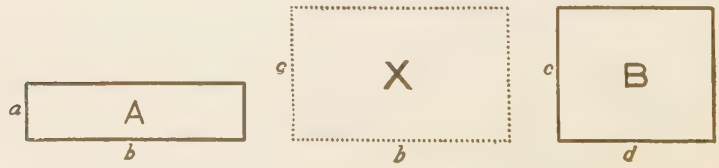
Also $\frac{AB}{ER}$ approaches $\frac{AB}{EF}$ as a limit.

$\therefore \frac{AC}{EG} = \frac{AB}{EF}$ (229). Q.E.D.

355. Corollary. **Two rectangles having equal bases are to each other as their altitudes.** (Explain.)

Proposition II. Theorem

356. **Any two rectangles are to each other as the products of their bases by their altitudes.**



Given: Rectangles A and B the altitudes of which are a and c, and the bases b and d, respectively.

To Prove: $A : B = a \cdot b : c \cdot d$.

Proof: Construct a third rectangle X, whose base is b and whose altitude is c.

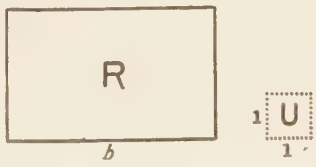
Then $\frac{A}{X} = \frac{a}{c}$ (355).

Also $\frac{X}{B} = \frac{b}{d}$ (354).

Multiplying, $\frac{A}{B} = \frac{a \cdot b}{c \cdot d}$ (Ax. 3). Q.E.D.

Proposition III. Theorem

357. The area of a rectangle is equal to the product of its base by its altitude.



Given: Rectangle R, with base b and altitude h.

To Prove: Area of $R = b \cdot h$.

Proof: Draw a square U, each side of which is a unit of length. This square is a unit of surface (352).

Now $\frac{R}{U} = \frac{b \cdot h}{1 \cdot 1} = b \cdot h$ (356).

But $\frac{R}{U} =$ the area of R (353).

$\therefore$ the area of $R = b \cdot h$ (Ax. 1).

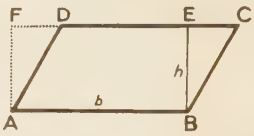
Q.E.D.

358. Corollary. The area of a square is equal to the square of its side. (357.)

Ex. I have enough material to build 1000 yards of fence. If I put this around a square field, how many square yards will the field contain? If I put it around a rectangular field that is four times as long as it is wide, how many square yards will the field contain?

Proposition IV. Theorem

359. The area of a parallelogram is equal to the product of its base by its altitude.



Given: $\square ABCD$, with base b and altitude h.

To Prove: Area of $ABCD = b \cdot h$.

Proof: From A and B, the extremities of the base, draw $\perp$s to the upper base meeting it in F and E respectively.

In rt. $\triangle ADF$ and BCE,

$AF = BE$ (124).

$AD = BC$ (124).

$\therefore \triangle ADF$ is congruent to $\triangle BCE$ (84).

Now from the whole figure subtract $\triangle ADF$ and the parallelogram $ABCD$ remains. And from the whole figure subtract $\triangle BCE$ and rectangle $ABEF$ remains.

$\therefore \square ABCD =$ rectangle $ABEF$ (Ax. 2).

But rectangle $ABEF = b \cdot h$ (357).

$\therefore \square ABCD = b \cdot h$ (Ax. 1).

Q.E.D.

360. Corollary. All parallelograms having equal bases and equal altitudes are equal in area.

361. Corollary. Two parallelograms having equal altitudes are to each other as their bases.

Proof: $P = b \cdot h$ and $P' = b' \cdot h$ (359).

Dividing, $\frac{P}{P'} = \frac{b \cdot h}{b' \cdot h} = \frac{b}{b'}$ (Ax. 3).

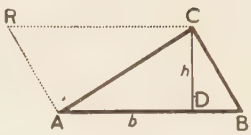
Q.E.D.

362. Corollary. Two parallelograms having equal bases are to each other as their altitudes. Proof: (?).

363. Corollary. Any two parallelograms are to each other as the products of their bases by their altitudes. Proof: (?).

Proposition V. Theorem

364. **The area of a triangle is equal to half the product of its base by its altitude.**



Given: $\triangle ABC$, with base b and altitude h.

To Prove: Area of $\triangle ABC = \frac{1}{2}\, b \cdot h$.

Proof: Through A draw $AR \parallel$ to BC and through C draw $CR \parallel$ to AB, meeting AR at R.

Now	$ABCR$ is a $\square$	(120).
The area of	$\square\ ABCR = b \cdot h$	(359).
Dividing by 2,	$\frac{1}{2}\ \square\ ABCR = \frac{1}{2}\, b \cdot h$	(Ax. 3).
Also	$\frac{1}{2}\ \square\ ABCR = \triangle\ ABC$	(126).
$\therefore$ the area of	$\triangle\ ABC = \frac{1}{2}\, b \cdot h$	(Ax. 1). Q.E.D.

365. Corollary. **A triangle having the same base and altitude as a parallelogram equals half the parallelogram.**

366. Corollary. **All triangles having equal bases and equal altitudes are equal in area.**

367. Corollary. **All triangles having the same base and whose vertices are in a line parallel to the base are equal.**

368. Corollary. **Two triangles having equal altitudes are to each other as their bases.**

Proof: $\triangle\ T = \frac{1}{2}\, b \cdot h$; and $\triangle\ T' = \frac{1}{2}\, b'h$ (364).

Dividing, $$\frac{\triangle\ T}{\triangle\ T'} = \frac{\frac{1}{2}\, bh}{\frac{1}{2}\, b'h} = \frac{b}{b'}$$ (Ax. 3).

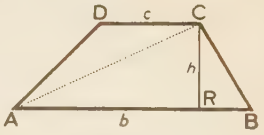
369. Corollary. **Two triangles having equal bases are to each other as their altitudes.** Proof: (?).

370. Corollary. **Any two triangles are to each other as the products of their bases by their altitudes.** Proof: (?).

371. Corollary. **The area of a right triangle is equal to half the product of the legs.** Proof: (?).

Proposition VI. Theorem

372. The area of a trapezoid is equal to half the product of the altitude by the sum of the bases.



Given: Trapezoid $ABCD$, with altitude h and bases b and c.

To Prove: Area $= \frac{1}{2}\, h \cdot (b + c)$.

Proof: Draw diagonal AC. The $\triangle ABC$ and ADC have the same altitude, h, and their bases are b and c, respectively.

Now	$\triangle ABC = \frac{1}{2}\, b \cdot h$	(364).
Also	$\triangle ADC = \frac{1}{2}\, c \cdot h$	(?).
Adding,	$\triangle ABC + \triangle ADC = \frac{1}{2}\, b \cdot h + \frac{1}{2}\, c \cdot h$	(Ax. 2).
That is,	trapezoid $ABCD = \frac{1}{2}\, h \cdot (b + c)$ (Ax. 6).	Q.E.D.

373. Corollary. The area of a trapezoid is equal to the product of the altitude by the median.

Proof: Area $ABCD = \frac{1}{2}\, h \cdot (b + c) = h \cdot \frac{1}{2}(b + c)$		(372).
But	$\frac{1}{2}(b + c) =$ median	(138).
Hence	area of trapezoid $ABCD = h \cdot m$ (Ax. 6).	Q.E.D.

Ex. 1. If one parallelogram has half the base and the same altitude as another, the area of the first is half the area of the second.

Ex. 2. If one parallelogram has half the base and half the altitude of another, its area is one fourth the area of the second.

Ex. 3. State and prove two analogous theorems about triangles.

Ex. 4. If a triangle has half the base and half the altitude of a parallelogram, the triangle is one eighth of the parallelogram.

Ex. 5. The area of a rhombus equals half the product of its diagonals.

Ex. 6. The diagonals of a parallelogram divide it into four triangles of equal areas.

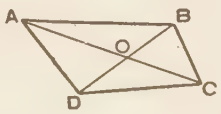
Ex. 7. The diagonals of a trapezoid divide it into four triangles, two of which are similar and the other two have equal areas.

Ex. 8. If a parallelogram has half the base and half the altitude of a triangle, its area is half the area of the triangle.

Ex. 9. The line joining the midpoints of two sides of a triangle forms a triangle whose area is one fourth the area of the original triangle.

Ex. 10. The line joining the midpoints of two adjacent sides of a parallelogram cuts off a triangle whose area is one eighth of the area of the parallelogram.

Ex. 11. If one diagonal of a quadrilateral bisects the other, it also divides the quadrilateral into two triangles having equal areas.



To Prove: $\triangle ABC = \triangle ADC$.

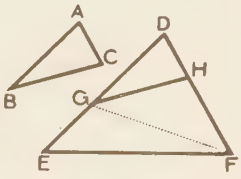
Ex. 12. Either diagonal of a trapezoid divides the figure into two triangles the ratio of which is equal to the ratio of the bases of the trapezoid.

Ex. 13. If, in triangle ABC, D and E are the midpoints of sides AB and AC respectively, $\triangle BCD = \triangle BEC$.

Ex. 14. If the diagonals of quadrilateral $ABCD$ meet at E, and $\triangle ABE$ is equal in area to $\triangle CDE$, the sides AD and BC are parallel.

Proposition VII. Theorem

374. If two triangles have an angle of one equal to an angle of the other, they are to each other as the products of the sides including the equal angles.



Given: $\triangle ABC$ and DEF, $\angle A = \angle D$.

To Prove: $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$.

Proof: Superpose $\triangle ABC$ upon $\triangle DEF$ so that the equal $\angle$s coincide and BC takes the position denoted by GH. Draw GF.

Now $\triangle DGH$ and DGF have the same altitude (a $\perp$ from G to DF), and $\triangle DGF$ and DEF have the same altitude (a $\perp$ from F to DE).

$\therefore \dfrac{\triangle DGH}{\triangle DGF} = \dfrac{DH}{DF}$ and $\dfrac{\triangle DGF}{\triangle DEF} = \dfrac{DG}{DE}$ (368).

Multiplying, $\dfrac{\triangle DGH}{\triangle DEF} = \dfrac{DG \cdot DH}{DE \cdot DF}$ (Ax. 3).

That is, $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$ (Ax. 6).

Q.E.D.

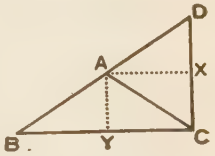
Ex. 1. If two triangles have an angle of one the supplement of an angle of the other, the triangles are to each other as the products of the sides including these angles.

Ex. 2. If two triangles of equal area have an angle of one equal to an angle of the other, the sides including these angles are reciprocally proportional.

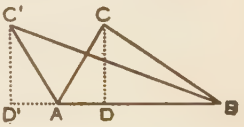
Ex. 3. Any two sides of a triangle are reciprocally proportional to the altitudes upon them.

Ex. 4. In triangles of equal area the bases and the altitudes upon them are reciprocally proportional.

Ex. 5. If two isosceles triangles have the legs of one equal to the legs of the other, and the vertex angle of the one the supplement of the vertex angle of the other, the triangles have equal areas.

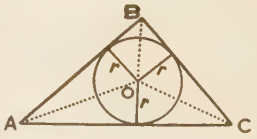


Ex. 6. Two triangles are equal in area if they have two sides of one equal to two sides of the other and the included angles supplementary.



Ex. 7. The area of a triangle is equal to half the perimeter of the triangle multiplied by the radius of the inscribed circle.

Ex. 8. The area of a polygon circumscribed about a circle is equal to half the product of the perimeter of the polygon by the radius of the circle.



Ex. 9. The line joining the midpoints of the bases of a trapezoid bisects the area of the trapezoid.

Ex. 10. Any line drawn through the midpoint of a diagonal of a parallelogram, intersecting two sides, bisects the area of the parallelogram.

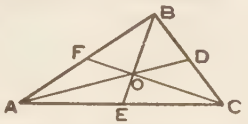
Ex. 11. The lines joining (in order) the midpoints of the sides of any quadrilateral form a parallelogram whose area is half the area of the quadrilateral.

Ex. 12. If any point within a parallelogram is joined to the four vertices, the sum of one pair of opposite triangles is equal to the sum of the other pair; that is, to half the parallelogram.

Ex. 13. Is a triangle bisected by an altitude? by the bisector of an angle? by a median? by the perpendicular bisector of a side? Give reasons.

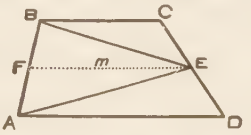
Ex. 14. If the three medians of a triangle are drawn, there are six pairs of triangles formed, one of each pair being double the other.

For instance, $\triangle AOB = 2 \triangle AOE$; etc.



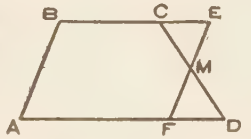
Ex. 15. If the midpoints of two sides of a triangle are joined to any point in the base, the quadrilateral formed is half the original triangle.

Ex. 16. If lines are drawn from the midpoint of one leg of a trapezoid to the ends of the other leg, the middle triangle thus formed is equal in area to half the trapezoid.



Ex. 17. The area of a trapezoid is equal to the product of one of the non-parallel sides, by the perpendicular upon it from the midpoint of the other.

Ex. 18. If through the midpoint of one of the non-parallel sides of a trapezoid a line is drawn parallel to the other side, the parallelogram formed is equal in area to the trapezoid.



Ex. 19. The sum of the three perpendiculars drawn to the three sides of an equilateral triangle from any point within is constant (being equal to the altitude of the triangle).

Proof: Join the point to the vertices. Set the sum of the areas of the three inner ▲ equal to the area of the whole △. Etc.

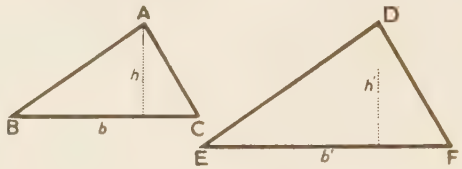
Historical Note. Sir Isaac Newton was born on Dec. 25, 1642, at Grantham, England. At an early age he exhibited a fondness and aptitude for mechanical contrivances, — windmills, water-clocks, kites and dials.

NEWTON

Later in his career he studied Descartes' geometry and was inspired with a love for all the mathematical studies. In the years 1665 and 1666 he made many important mathematical discoveries. He invented improvements for both the telescope and microscope, and discovered the existence of the spectrum. The study of Kepler's laws of motion resulted in Newton's discovery of the law of gravitation, which he discussed with such contemporaries as Sir Christopher Wren, Hooke, and the astronomer Halley. He also discovered the binomial theorem, which was inscribed on his tomb when he died in 1727. He has always been honored as the greatest mathematician of all time.

Proposition VIII. Theorem

375. Two similar triangles are to each other as the squares of any two homologous sides.



Given: Similar $\triangle ABC$ and DEF.

To Prove: $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{\overline{AB}^2}{\overline{DE}^2} = \dfrac{\overline{AC}^2}{\overline{DF}^2} = \dfrac{\overline{BC}^2}{\overline{EF}^2}$.

Proof: Denote a pair of homologous altitudes by h and h', and the corresponding bases by b and b'.

Now $$\frac{\triangle ABC}{\triangle DEF} = \frac{b}{b'} \cdot \frac{h}{h'} = \frac{b}{b'} \cdot \frac{h}{h'} \qquad (370).$$

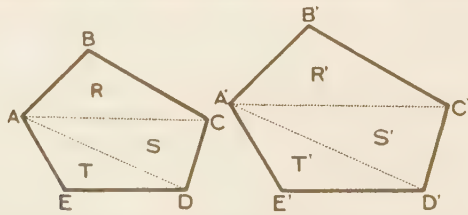
But $$\frac{h}{h'} = \frac{b}{b'} \qquad (311).$$

Substituting, $$\frac{\triangle ABC}{\triangle DEF} = \frac{b}{b'} \cdot \frac{b}{b'} = \frac{b^2}{b'^2} \qquad (\text{Ax. } 6).$$

That is, $$\frac{\triangle ABC}{\triangle DEF} = \frac{\overline{BC}^2}{\overline{EF}^2} \text{ or } = \frac{\overline{AB}^2}{\overline{DE}^2} \text{ or } = \frac{\overline{AC}^2}{\overline{DF}^2}. \qquad \text{Q.E.D.}$$

Proposition IX. Theorem

376. Two similar polygons are to each other as the squares of any two homologous sides and as the squares of their perimeters.



Given: Similar polygons $ABCDE$ and $A'B'C'D'E'$, with perimeters P and P' respectively.

To Prove: $\dfrac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \dfrac{\overline{AB}^2}{\overline{A'B'}^2} = \text{etc., and} = \dfrac{P^2}{P'^2}$.

Proof: Draw from homologous vertices, A and A', all the pairs of homologous diagonals, dividing the polygons into $\triangle$. These $\triangle$ are similar in pairs. (318).

$$\frac{\triangle R}{\triangle R'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (375).$$

$$\frac{\triangle S}{\triangle S'} = \frac{\overline{CD}^2}{\overline{C'D'}^2} \qquad (?).$$

$$\frac{\triangle T}{\triangle T'} = \frac{\overline{DE}^2}{\overline{D'E'}^2} \qquad (?).$$

But $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} \qquad (313).$$

$$\therefore \frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{\overline{BC}^2}{\overline{B'C'}^2} = \frac{\overline{CD}^2}{\overline{C'D'}^2} = \frac{\overline{DE}^2}{\overline{D'E'}^2} \qquad (287).$$

$$\therefore \frac{\triangle R}{\triangle R'} = \frac{\triangle S}{\triangle S'} = \frac{\triangle T}{\triangle T'} \qquad (\text{Ax. } 1).$$

$$\therefore \frac{\triangle R + \triangle S + \triangle T}{\triangle R' + \triangle S' + \triangle T'} = \frac{\triangle R}{\triangle R'} \qquad (291).$$

Substituting, $$\frac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \frac{\triangle R}{\triangle R'} \qquad (\text{Ax. } 6).$$

But $$\frac{\triangle R}{\triangle R'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (\text{Above}).$$

$$\therefore \frac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \frac{\overline{AB}^2}{\overline{A'B'}^2} \qquad (\text{Ax. } 6).$$

Q.E.D.

Also $$P : P' = AB : A'B' = \text{etc.} \qquad (317).$$

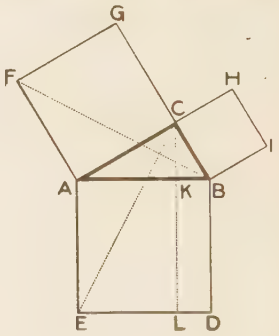
$$\therefore P^2 : P'^2 = \overline{AB}^2 : \overline{A'B'}^2 = \text{etc.} \qquad (287).$$

$$\therefore \frac{\text{Polygon } ABCDE}{\text{Polygon } A'B'C'D'E'} = \frac{P^2}{P'^2}. \qquad (\text{Ax. } 1).$$

Q.E.D.

Proposition X. Theorem

377. The square described upon the hypotenuse of a right triangle is equal in area to the sum of the squares described upon the legs.



Given: (?).

To Prove: (?).

Proof: Draw $CL \perp$ to AB, meeting AB at K and ED at L. Draw BF and CE.

Now	$\measuredangle ACB$, ACG, and BCH are all rt. $\measuredangle$.	(Hyp.)
Hence	ACH and BCG are straight lines.	(45.)
Also	$AELK$ and $BDLK$ are rectangles.	(Def.)
In	$\triangle ABF$ and ACE, $AB = AE$, $AF = AC$	(122).

$$\angle BAF = \angle CAE$$

(Each is composed of a rt. $\angle$ plus $\angle CAB$).

$$\therefore \triangle ABF \cong \triangle ACE \qquad (52).$$

Also $\triangle ABF$ and square AG have the same base, AF, and the same altitude, AC.

$$\therefore \text{ square } AG = 2 \cdot \triangle ABF \qquad (365).$$

Similarly, $\quad$ rectangle $AKLE = 2 \cdot \triangle ACE \qquad (365).$

$$\therefore \text{ rectangle } AKLE = \text{square } AG \qquad (\text{Ax. } 1).$$

By drawing AI and CD, it may be proved in the same manner that rectangle $BDLK$ = square BH.

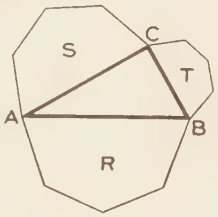
By adding, square AD = square AG + square BH (Ax. 2).

Q.E.D.

378. Corollary. The square described upon one of the legs of a right triangle is equal in area to the square described upon the hypotenuse minus the square described upon the other leg.

Proposition XI. Theorem

379. If the three sides of a right triangle are the homologous sides of three similar polygons, the polygon described upon the hypotenuse is equal in area to the sum of the two polygons described upon the legs.



Given: (?). **To Prove:** (?).

Proof: $\frac{S}{R} = \frac{\overline{AC}^2}{\overline{AB}^2}$ (376).

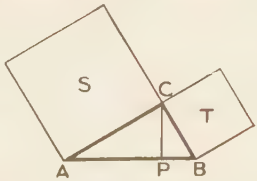
And $\frac{T}{R} = \frac{\overline{BC}^2}{\overline{AB}^2}$ (?).

Adding, $\frac{S+T}{R} = \frac{\overline{AC}^2 + \overline{BC}^2}{\overline{AB}^2} = \frac{\overline{AB}^2}{\overline{AB}^2} = 1.$ (Ax. 2; 334.)

Clearing of fractions, $R = S + T$ (Ax. 3). Q.E.D.

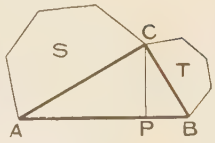
380. Corollary. If the three sides of a right triangle are the homologous sides of three similar polygons, the polygon described upon one of the legs is equal in area to the polygon described upon the hypotenuse minus the polygon described upon the other leg.

381. Corollary. The two squares described upon the legs of a right triangle are to each other as the projections of the legs upon the hypotenuse.



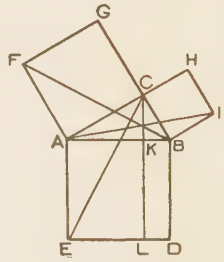
Proof: Square $S = \overline{AC}^2$ Square $T = \overline{BC}^2$ (?).

$\therefore \frac{\text{Square } S}{\text{Square } T} = \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \cdot AP}{AB \cdot BP} = \frac{AP}{BP}$ (Ax. 3; 333).



382. Corollary. If two similar polygons are described upon the legs of a right triangle as homologous sides, they are to each other as the projections of the legs upon the hypotenuse.

Proof: $\frac{\text{Polygon } S}{\text{Polygon } T} = \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \cdot AP}{AB \cdot BP} = \frac{AP}{BP}$ (376; 333).



Ex. 1. In the figure of 377, prove that:

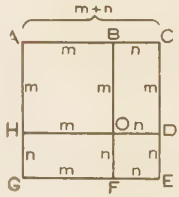
(*i*) Points I, C, and F are in a straight line.

(*ii*) CE and BF are perpendicular.

(*iii*) AG and BH are parallel.

(*iv*) $\triangle AEF = \triangle CGH = \triangle BDI = \triangle ABC$.

Ex. 2. The sum of the squares described upon the four segments of two perpendicular chords in a circle is equal to the square described upon the diameter. (Fig. is on page 185.)

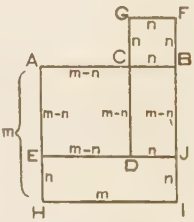


Ex. 3. The square described upon the sum of two lines is equal to the sum of the squares described upon the two lines, plus twice the rectangle of these lines.

To Prove: Square $AE = m^2 + n^2 + 2\,mn$.

Ex. 4. The square described upon the difference of two lines is equal to the sum of the squares described upon the two lines minus twice the rectangle of these lines.

To Prove: Square $AD = m^2 + n^2 - 2\,mn$.



Ex. 5. A and B are the extremities of a diameter of a circle; C and D are the points of intersection of any third tangent to this circle with the tangents at A and B respectively. Prove that the area of $ABDC$ is equal to $\frac{1}{2}\,AB \cdot CD$.

Ex. 6. If the four points midway between the center and vertices of a parallelogram are joined in order, a parallelogram is formed similar to the original parallelogram; its perimeter is half of the perimeter of the original figure; and its area is one quarter of the area of the original figure.

Ex. 7. If two triangles of equal area have the same base and lie on opposite sides of it, the line joining their vertices is bisected by the line of the base.

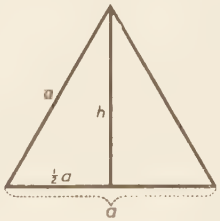
Ex. 8. What part of a right triangle is the quadrilateral which is cut from the triangle by a line joining the midpoints of the legs?

Ex. 9. From M, a vertex of parallelogram $LMNO$, a line MPX is drawn meeting NO at P and LO produced, at X. LP and NX are also drawn. Prove that triangles LOP and XNP are equal in area.

FORMULAS

Proposition XII. Problem

383. To derive the formulas for the altitude and the area of an equilateral triangle, in terms of its side.



Solution: Let each side $= a$, and altitude $= h$. $h^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$ (335).

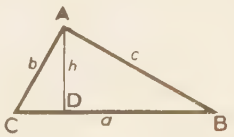
I. $\therefore h = \frac{a}{2}\sqrt{3}$.

$\text{Area} = \frac{1}{2}a \cdot h = \frac{1}{2}a \cdot \frac{a}{2}\sqrt{3}$ (364).

II. $\therefore$ **Area** $= \frac{a^2\sqrt{3}}{4}$.

Proposition XIII. Problem

384. To derive a formula for the area of a triangle in terms of its sides.



Given: $\triangle ABC$, having sides a, b, c.

Required: To derive a formula for its area, containing only a, b, and c.

Solution: Draw altitude AD.

Now $CD = {}_bp_a = \frac{a^2 + b^2 - c^2}{2a}$ (339).

Also $\overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2$ (335).

$\therefore h^2 = b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2$ (Ax. 6).

Hence $h^2 = \left\{b + \frac{a^2+b^2-c^2}{2a}\right\}\left\{b - \frac{a^2+b^2-c^2}{2a}\right\}$ (by factoring).

Also $h^2 = \frac{2ab + a^2 + b^2 - c^2}{2a} \cdot \frac{2ab - a^2 - b^2 + c^2}{2a}$.

$$\therefore h = \sqrt{\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2}}.$$

Let $a + b + c = 2s$.

Then it is evident that $a + b - c = 2(s - c)$
and $a - b + c = 2(s - b)$ and $-a + b + c = 2(s - a)$.

Substituting above, $h = \sqrt{\frac{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)}{4a^2}}$

$$= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}.$$

Now area $\triangle = \frac{1}{2}a \cdot h = \frac{a}{2} \cdot \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore \textbf{Area of } \triangle = \sqrt{s(s-a)(s-b)(s-c)}.$$

Exercise. Find the area of a triangle whose sides are 17, 25, 28.
Here, $a = 17$, $b = 25$, $c = 28$, $s = 35$, $s - a = 18$, $s - b = 10$, $s - c = 7$.
Area $= \sqrt{35 \cdot 18 \cdot 10 \cdot 7} = \sqrt{7^2 \cdot 5^2 \cdot 2^2 \cdot 3^2} = 210$.

Ex. 1. Find the area of the triangle whose sides are 7, 10, 11.

Ex. 2. Find the area of the triangle whose sides are 8, 15, 17.

Ex. 3. Find the area of the equilateral triangle whose side is 8.

Ex. 4. Find the side of the equilateral triangle whose area is $121\sqrt{3}$.

Ex. 5. Find the area of the equilateral triangle whose altitude is 10.

Proposition XIV. Problem

385. To derive formulas for the altitudes of a triangle in terms of the three sides.

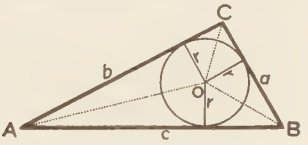
Solution: Area $= \frac{1}{2}ah_a = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\therefore h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}a}.$$

Similarly, $h_b = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}b}$; $h_c = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}c}$.

Proposition XV. Problem

386. To derive the formula for the radius of the circle inscribed in a triangle, in terms of the sides of the triangle.



Solution:

$$\left.\begin{array}{l}\text{Area of } \triangle AOB = \frac{1}{2}\, c \cdot r \\ \text{area of } \triangle AOC = \frac{1}{2}\, b \cdot r \\ \text{area of } \triangle BOC = \frac{1}{2}\, a \cdot r\end{array}\right\} \quad (?).$$

Adding, $\text{area of } \triangle ABC = \frac{1}{2}(a+b+c)r = sr.$

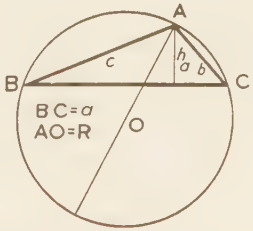
[Because $\frac{1}{2}(a+b+c) = s.$]

Hence $$r = \frac{\text{area of } \triangle ABC}{s}.$$

$$\therefore r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}.$$

Proposition XVI. Problem

387. To derive the formula for the radius of the circle circumscribed about a triangle, in terms of the sides of the triangle.



Solution:

$$2R \cdot h_a = b \cdot c \qquad (328).$$

$$\therefore R = \frac{b \cdot c}{2h_a}.$$

$$h_a = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{\frac{1}{2}\, a} \qquad (385).$$

$$\therefore R = \frac{a \cdot b \cdot c}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

Ex. 1. Find the radius of the circle inscribed in, and the radius of the circle circumscribed about, the triangle whose sides are 17, 25, 28.

Ex. 2. Find for triangle whose sides are 11, 14, 17, the radii of the inscribed and circumscribed circles.

ORIGINAL EXERCISES (NUMERICAL)

1. The base of a parallelogram is 2 ft. 6 in. and its altitude is 1 ft. 4 in. Find the area. Find the side of a square of equal area.

2. The area of a rectangle is 540 sq. m. and its altitude is 15 m. Find its base and its diagonal.

3. The base of a rectangle is 3 ft. 4 in. and its diagonal is 3 ft. 5 in. Find its area.

4. The bases of a trapezoid are 2 ft. 1 in., and 3 ft. 4 in., and the altitude is 1 ft. 2 in. Find the area.

5. The area of a trapezoid is 736 sq. in. and its bases are 3 ft. and 4 ft. 8 in. Find the altitude.

6. The area of a certain triangle whose base is 40 rd. is 3.2 A. Find the area of a similar triangle whose base is 10 rd. Find the altitudes of these triangles.

7. The base of a certain triangle is 20 cm. Find the base of a similar triangle four times as large; of one five times as large; twice as large; half as large; one ninth as large.

8. The altitude of a certain triangle is 12 and its area is 100. Find the altitude of a similar triangle three times as large. Find the base of a similar triangle seven times as large. Find the altitude and the base of a similar triangle one third as large.

9. The area of a polygon is 216 sq. m. and its shortest side is 8 m. Find the area of a similar polygon whose shortest side is 10 m. Find the shortest side of a similar polygon four times as large; one tenth as large.

10. If the longest side of a polygon whose area is 567 is 14, what is the area of a similar polygon whose longest side is 12? of another whose longest side is 21?

11. Find the area of an equilateral triangle whose sides are each 6 in.; of another whose sides are each $10\sqrt{3}$ ft.

12. Find the area of an equilateral triangle whose altitude is 4 in.; of another whose altitude is 18 dm.

13. The area of an equilateral triangle is $64\sqrt{3}$. Find its side and its altitude.

14. The area of an equilateral triangle is 90 sq. m. Find its altitude.

15. Find the side of an equilateral triangle whose area is equal to a square whose side is 15 ft.

16. The equal sides of an isosceles triangle are each 17 in. and the base is 16 in. Find the area.

17. Find the area of an isosceles right triangle whose hypotenuse is 2 ft. 6 in.

18. Find the area of a square whose diagonal is 20 m.

19. There are two equilateral triangles whose sides are 33 and 56 respectively. Find the side of the equilateral triangle equal to their sum. Find the side of the equilateral triangle equal to their difference.

20. There are two similar polygons two of whose homologous sides are 24 and 70. Find the side of a third similar polygon equal to their sum; the side of a similar polygon equal to their difference.

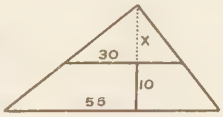
21. What is the area of the right triangle whose hypotenuse is 29 cm. and whose short leg is 20 cm.?

22. The base of a triangle is three times the base of a triangle equal in area. What is the ratio of their altitudes?

23. The bases of a trapezoid are 56 ft. and 44 ft. and the non-parallel sides are each 10 ft. Find its area. Also find the diagonal of a square of equal area.

24. The base of a triangle is 80 m., and its altitude is 8 m. Find the area of the triangle cut off by a line parallel to the base and at a distance of 3 m. from it. Find the area of another triangle, cut off by a line parallel to the base and 6 m. from it.

25. The bases of a trapezoid are 30 and 55, and its altitude is 10. If the non-parallel sides are produced till they meet, find the area of the less triangle formed.



[The $\triangle$s are similar. $\therefore 30:55 = x:x+10$. Etc.]

26. The diagonals of a rhombus are 2 ft. and 70 in. Find the area; the perimeter; the altitude.

27. The altitude (h) of a triangle is increased by n and the base (b) is diminished by x so that the area remains unchanged. Find x.

28. The projections of the legs of a right triangle upon the hypotenuse are 8 and 18. Find the area of the triangle.

29. In triangle ABC, AB is 5, BC is 8, and AB is produced to P, making $BP = 6$. BC is produced (through B) to Q and PQ is drawn so that triangle BPQ is equal in area to triangle ABC. Find the length of BQ.

30. The angle C of triangle ABC is right; $AC = 5$; $BC = 12$. BA is produced through A, to D making $AD = 4$; CA is produced through A, to E so that triangle AED is equal in area to triangle ABC. Find AE.

31. Find the area of a square inscribed in a circle whose radius is 6.

32. Find the side of an equilateral triangle whose area is $25\sqrt{3}$.

33. Two sides of a triangle are 12 and 18. What is the ratio of the two triangles formed by the bisector of the angle between these sides?

34. The perimeter of a rectangle is 28 m. and one side is 5 m. Find the area.

35. The perimeter of a polygon is 5 ft. and the radius of the inscribed circle is 5 in. Find the area of the polygon.

In the following triangles, find the area, the three altitudes, the radius of the inscribed circle, the radius of circumscribed circle:

36. $a = 13$, $b = 14$, $c = 15$.

37. $a = 15$, $b = 41$, $c = 52$.

38. 20, 37, 51.

39. 140, 143, 157.

40. The sides of a triangle are 15, 41, 52. Find the areas of the two triangles into which this triangle is divided by the bisector of the largest angle.

41. Find the area of the quadrilateral $ABCD$ if $AB = 78$ m., $BC = 104$ m., $CD = 50$ m., $AD = 120$ m., and $AC = 130$ m.

42. One diagonal of a rhombus is $\frac{8}{15}$ of the other and the difference of the diagonals is 14. Find the area and the perimeter of the rhombus.

43. A trapezoid is composed of a rhombus and an equilateral triangle. Each side of each figure is 16 inches. Find the area of the trapezoid.

44. Find the side of an equilateral triangle equal in area to the square whose diagonal is $15\sqrt{2}$.

45. Which of the figures in Ex. 44 has the smaller perimeter?

46. In a triangle whose base is 20 and whose altitude is 12, a line is drawn parallel to the base, bisecting the area of the triangle. Find the distance from the base to this parallel.

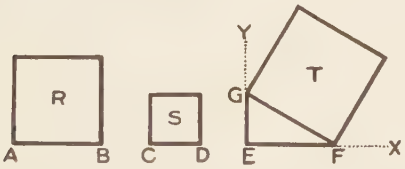
47. Two lines are drawn parallel to the base of a triangle whose base is 30 and altitude 18. These lines divide the area of the triangle into three equal parts. Find their distances from the vertex.

48. Around a rectangular lawn 30 yards × 20 yards is a drive 16 feet wide. How many square yards are there in the drive?

CONSTRUCTION PROBLEMS

Proposition XVII. Problem

388. To construct a square equal to the sum of two squares.



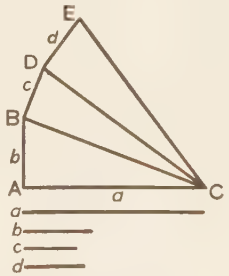
Given: (?). **Required:** (?). **Construction:** Construct a rt. $\angle E$, with sides EX and EY. On EX take $EF =$ to AB; on EY take $EG =$ to CD. Draw FG. On FG construct square T.

Statement: $T = R + S$. Q.E.F.

Proof: $T = R + S$ (377). Q.E.D.

Proposition XVIII. Problem

389. To construct a square equal to the sum of several squares.



Given: Squares whose sides are a, b, c, d.

Required: To construct a square $=$ to $a^2 + b^2 + c^2 + d^2$.

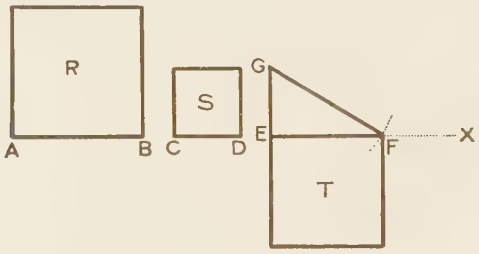
Construction: Construct a rt. $\angle$ whose sides are equal to a and b. Draw hypotenuse BC. At B erect a $\perp =$ to c and draw hypotenuse DC. At D erect a $\perp =$ to d, etc.

Statement: The square constructed on $EC =$ the sum of the several given squares. Q.E.F.

$$\overline{EC}^2 = \overline{DC}^2 + d^2 \quad (334).$$
$$= \overline{BC}^2 + c^2 + d^2 \quad (?).$$
$$= a^2 + b^2 + c^2 + d^2 \quad (?). \quad \text{Q.E.D.}$$

Proposition XIX. Problem

390. To construct a square equal to the difference of two given squares.



Given: (?). **Required:** (?).

Construction: At one end of indefinite line, EX, erect $EG \perp$ to EX and $=$ to CD (a side of the **smaller** square, S). Using G as center and AB as radius, describe arc intersecting EX at F. Draw GF. On EF construct square T.

Statement: $T = R - S$. Q.E.F. **Proof:** (?).

391. To construct a polygon similar to two given similar polygons and equal to their sum.

Construction: Like 388. **Proof:** (379).

392. To construct a polygon similar to two given similar polygons and equal to their difference.

Construction: Like 390. **Proof:** (380).

Ex. 1. Construct a right triangle whose area shall equal the area of a given square.

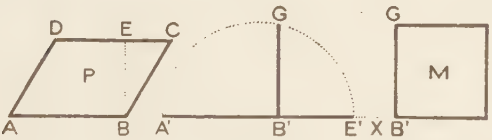
Ex. 2. Construct an isosceles triangle whose area shall equal the area of a given square.

Ex. 3. Construct an isosceles triangle equal in area to a given right triangle.

Ex. 4. Construct an isosceles triangle equal in area to any given triangle.

Proposition XX. Problem

393. To construct a square equal in area to a given parallelogram.



Given: (?). **Required:** (?) **Construction:** Construct a mean proportional between the base, AB, and the altitude, BE; on this mean proportional, $B'G$, construct a square, M.

Statement:	Square M = parallelogram P.	Q.E.F.
Proof:	$AB : B'G = B'G : BE$	(Const.).
	$\therefore \overline{B'G}^2 = AB \cdot BE$	(280).
But	$\overline{B'G}^2 =$ square M	(358).
And	$AB \cdot BE = \square P$	(359).
	$\therefore$ square $M = \square P$	(Ax. 1).
		Q.E.D.

394. To construct a square equal in area to a given triangle:

Construct a mean proportional between half the base and the altitude, and proceed as in 393.

Ex. 1. Construct a square equal in area to a given right triangle.

Ex. 2. Construct a square equal in area to the sum of any two triangles; equal to their difference.

Ex. 3. Construct a square equal in area to the sum of two parallelograms; equal to their difference.

Ex. 4. Construct a right triangle equal in area to a given parallelogram, and with the same base.

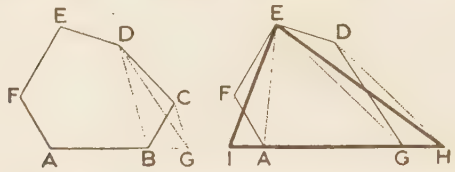
Ex. 5. Construct an isosceles triangle equal in area to a given parallelogram, and with the same base.

Ex. 6. Construct on the same base as a square, an isosceles triangle equal in area to the square.

Ex. 7. Construct a right triangle equal in area to the sum of two given squares.

Proposition XXI. Problem

395. To construct a triangle equal to a given polygon.



Given: Polygon AD. **Required:** To construct a $\triangle$ = to AD.

Construction: Draw a diagonal, BD, connecting any vertex (B) to the next but one (D). From the vertex between these (C), draw $CG \parallel$ to BD, meeting AB prolonged, at G. Draw DG. Repeat (2d figure) by drawing EG and $DH \parallel$ to EG, then EH. Repeat again by drawing AE, $FI \parallel$ to AE, then EI.

Statement: $\triangle IHE =$ polygon $ABCDEF$. Q.E.F.

Proof: In first figure, $\triangle BGD = \triangle BCD$ (367).

Add polygon $ABDEF$ = polygon $ABDEF$

$\therefore$ pentagon $AGDEF$ = polygon $ABCDEF$ (Ax. 2).

Also, in second figure, $\triangle GHE = \triangle GDE$ (367).

Add polygon $AGEF$ = polygon $AGEF$

$\therefore$ quadrilateral $AHEF$ = pentagon $AGDEF$ (Ax. 2).

Again, $\triangle AIE = \triangle AFE$ (367).

Add $\triangle AHE = \triangle AHE$

$\therefore$ $\triangle IHE =$ quad. $AHEF$ (Ax. 2).

Hence $\triangle IHE =$ polygon $ABCDEF$ (Ax. 1).

Q.E.D.

396. To construct a square equal to a given polygon.

First, construct a $\triangle$ = to the polygon (by 395).

Second, construct a square = to the $\triangle$ (by 394).

Ex. 1. Tell how to construct, by two methods, a square equal to a parallelogram.

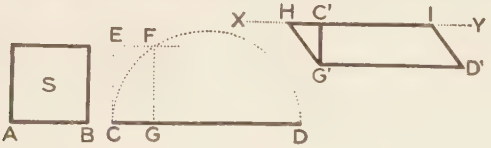
Ex. 2. How can rectilinear figures be added into a single figure?

Proposition XXII. Problem

397. To construct a parallelogram (or a rectangle) equal to a given square, and having:

I. The sum of its base and altitude equal to a given line.

II. The difference of its base and altitude equal to a given line.



I. Given: Square S and line CD.

Required: To construct a $\square$ = to S; base + altitude = CD.

Construction: On CD as a diameter describe a semicircle. At C erect $CE \perp$ to CD and = to AB. Through E draw $EF \parallel$ to CD, meeting the circumference at F. Draw $FG \perp$ to CD. Take $G'D' =$ to GD and draw $XY \parallel$ to $G'D'$ at the distance from it = to CG. On XY take $HI = GD$. Draw HG' and ID'.

Statement: $\square\ G'D'IH = S$, and base + alt. = CD. Q.E.F.

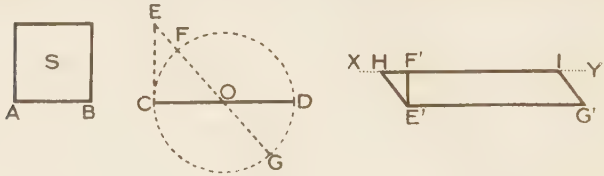
Proof:	$G'D'IH$ is a $\square$	(129).
Also	$GD \cdot GC = \overline{FG}^2$	(332).
But	$GD \cdot GC = \square\ G'D'IH$	(359).
And	$\overline{FG}^2 = \overline{EC}^2 = \overline{AB}^2 =$ square S	(124, 358).
Substituting,	$\square\ G'D'IH =$ square S	(Ax. 6).
Also	$G'D' + G'C' = CD$ (Ax. 4).	Q.E.D.

Historical Note. The following extract from Nicolay and Hay's *Life of Abraham Lincoln* should be of interest to the student:

"His wider knowledge of men and things had shown him a certain lack in himself of the power of close and sustained reasoning. To remedy this defect, he applied himself, after his return from Congress, to such works upon logic and mathematics as he fancied would be serviceable. Devoting himself with dogged energy to the task in hand, he soon learned by heart six books of the propositions of Euclid, and he retained through life an intimate knowledge of the principles they contain."

II. **Given:** Square S and line CD.

Required: To construct a $\square$ = to S; base − altitude = to CD.



Construction: On CD as diameter, describe a $\odot$, O. At C erect $CE \perp$ to CD and = to AB. Draw $EFOG$ meeting $\odot$ at F and G. Take $E'G'$ = to EG and draw $XY \parallel$ to $E'G'$ at a distance from it = to EF. On XY take HI = to EG. Draw HE' and IG'.

Statement: $\square\ E'G'IH = S$, and base − alt. = CD. Q.E.F.

Proof:

$E'G'IH$ is a $\square$ (129).

EC is tangent to $\odot\ O$ (202).

$\therefore\ EG \cdot EF = \overline{EC}^2$ (324).

But $EG \cdot EF = \square\ E'G'IH$ (359).

And $\overline{EC}^2 = \overline{AB}^2 =$ square S (358).

$\therefore\ \square\ E'G'IH =$ square S (Ax. 6).

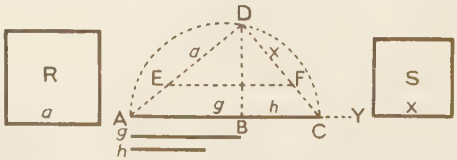
Also $E'G' - E'F' = FG = CD$ (190). Q.E.D.

398. To find two lines whose product is given:

I. If their sum is also given.
II. If their difference is also given. } [The same as 397.]

Proposition XXIII. Problem

399. To construct a square having a given ratio to a given square.



Given: Square R, and lines g and h.

Required: To construct a square such that it (the unknown square) : square $R = h : g$.

Construction: On an indefinite line AY take $AB =$ to g, and $BC =$ to h. On AC as diameter describe a semicircle. At B erect $BD \perp$ to AC, meeting arc at D. Draw AD and CD. On AD take $DE =$ to a, and draw $EF \parallel$ to AC, meeting DC at F. Using $DF = x$, as a side, construct square S.

Statement: $S : R = h : g$. Q.E.F.

Proof: $\angle ADC$ is a rt. $\angle$ (240).

$$\therefore \frac{\overline{CD}^2}{\overline{AD}^2} = \frac{h}{g} \quad (381).$$

But $$\frac{x}{a} = \frac{CD}{AD} \quad (294).$$

$$\therefore \frac{x^2}{a^2} = \frac{\overline{CD}^2}{\overline{AD}^2} \quad (287).$$

$$\therefore \frac{x^2}{a^2} = \frac{h}{g} \quad (\text{Ax. } 1).$$

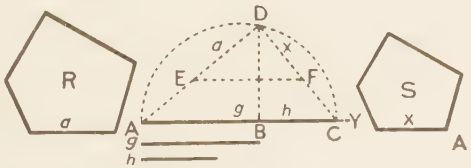
But $x^2 =$ square S, and $a^2=$ square R (358).

Substituting, square S : square $R = h : g$ (Ax. 6).

Q.E.D.

Proposition XXIV. Problem.

400. To construct a polygon similar to a given polygon and having a given ratio to it.

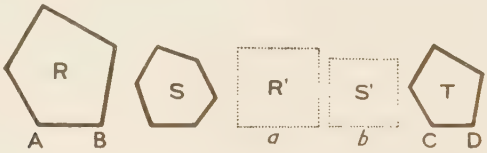


Given: (?). **Required:** (?). **Construction** and **Statement** are the same as for Proposition XXIII.

Proof: Very much like the proof of Proposition XXIII.

Proposition XXV. Problem

401. To construct a polygon similar to one given polygon and equal in area to another.



Given: Polygons R and S. **Required:** (?).

Construction: Construct squares $R' = R$, and $S' = S$ (by 396). Find a fourth proportional to a, b, and AB. This is CD. Upon CD, homologous to AB, construct T similar to R.

Statement: $T = S$. Q.E.F.

Proof:

$$\frac{R}{T} = \frac{\overline{AB}^2}{\overline{CD}^2} \quad \text{(376)}.$$

$$\frac{a}{b} = \frac{AB}{CD} \quad \text{(Const.)}.$$

$$\therefore \frac{a^2}{b^2} = \frac{\overline{AB}^2}{\overline{CD}^2} \quad \text{(287)}.$$

$$\therefore \frac{R}{T} = \frac{a^2}{b^2} \quad \text{(Ax. 1)}.$$

Now $a^2 = R' = R$; $b^2 = S' = S$ (358 & Const.).

Substituting, $\frac{R}{T} = \frac{R}{S}$ (Ax. 6).

$$\therefore T = S \quad \text{(Ax. 3)}.$$

Q.E.D.

Note. By means of this proposition we are able to maintain the size of a rectilinear figure, but change its shape to any desired form. Thus we can construct an equilateral triangle equal to a given square. The pupil will explain.

ORIGINAL CONSTRUCTIONS

It is required:

1. To construct a right triangle equal to a given parallelogram.

2. To construct a square equal to the sum of two given parallelograms.

3. To construct a square equal to the difference of two given parallelograms.

4. To construct a square equal to the sum of several given right triangles.

5. To construct a square equal to the sum of several given parallelograms.

6. To construct a square equal to the sum of several given triangles.

7. To construct a square equal to the sum of several given polygons.

8. To construct a square equal to the difference of two given polygons.

9. To construct a square equal to three times a given square; seven times a given square.

10. To construct a right triangle equal to the sum of several given triangles.

11. To construct a right triangle equal to the difference of any two given triangles; of any two given parallelograms.

12. To construct a square equal to a given trapezoid; equal to a given trapezium.

13. To construct a square equal to a given hexagon.

14. To construct a rectangle equal to a given triangle, having given its perimeter.

15. To construct an isosceles right triangle equal to a given triangle.

16. To construct a square equal to a given rhombus.

17. To construct a rectangle equal to a given trapezium, and having its perimeter given.

18. To find a line whose length shall be $\sqrt{2}$ units. [See 388.]

19. To find a line whose length shall be $\sqrt{3}$ units.

20. To find a line whose length shall be $\sqrt{11}$ units.

21. To find a line whose length shall be $\sqrt{7}$ units.

22. To find a line whose length shall be $\sqrt{30}$ units.

23. To construct a square which shall be $\frac{3}{5}$ of a given square.

24. To construct a square which shall be $\frac{4}{5}$ of a given square.

25. To construct a polygon which shall be $\frac{2}{7}$ of a given polygon, and similar to it.

26. To construct a square which shall have to a given square the ratio $\sqrt{3}:4$; the ratio $4:\sqrt{3}$.

27. To draw through a given point, within a parallelogram, a line which shall bisect the parallelogram.

28. To construct a rectangle equal to a given trapezoid, having given the difference of its base and altitude.

29. To construct a triangle similar to two given similar triangles and equal to their sum.

30. To construct a triangle similar to a given triangle and equal to a given square. [See 401.]

31. To construct a triangle similar to a given triangle and equal to a given parallelogram.

32. To construct a square having twice the area of a given square. [Two methods.]

33. To construct a square having $3\frac{1}{2}$ times the area of a given square.

34. To construct an isosceles triangle equal to a given triangle and upon the same base.

35. To construct a triangle equal to a given triangle, having the same base, and also having a given angle adjoining this base.

36. To construct a parallelogram equal to a given parallelogram having the same base and also having a given angle adjoining the base.

37. To draw a line that shall be perpendicular to the bases of a parallelogram and that shall bisect the parallelogram.

38. To construct an equilateral triangle equal to a given triangle. [See 401.]

39. To trisect (divide into three equal parts) the area of a triangle, by lines drawn from one vertex.

40. To construct a square equal to $\frac{2}{3}$ of a given pentagon.

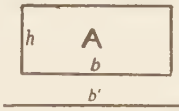
41. To construct an isosceles trapezoid equal to a given trapezoid.

42. To construct an equilateral triangle equal to the sum of two given equilateral triangles.

43. To construct an equilateral triangle equal to the difference of two given equilateral triangles.

44. To construct upon a given base a rectangle that shall be equal to a given rectangle.

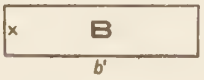
Analysis: Let us call the unknown altitude x. Then $b \cdot h = b' \cdot x$ (?). Hence, $b' : b = h : x$ (?).



That is, the *unknown altitude* is a fourth proportional to *the given base*, *the base of the given rectangle*, and *the altitude of the given rectangle.*

Construction: Find a fourth proportional, x, to b', b and h. Construct a rectangle having base $= b'$ and alt. $= x$.

Statement: This rectangle, $B = A$.



Proof: $b' : b = h : x$ (Const.). $\therefore b'x = bh$ (?). But $b'x =$ the area of B (?), etc.

45. To construct a rectangle that shall have a given altitude and be equal to a given rectangle.

46. To construct a triangle upon a given base that shall be equal to a given triangle.

47. To construct a triangle that shall have a given altitude and be equal to a given triangle.

48. To construct a rectangle that shall have a given base, and shall be equal to a given triangle.

49. To construct a triangle that shall have a given base, and be equal to a given rectangle.

50. To construct a triangle that shall have a given base, and be equal to a given polygon.

51. To construct the problems 45, 46, 47, 48, 49, substituting "parallelogram" in each case for the figure to be constructed.

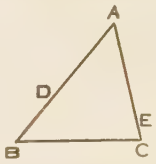
52. To construct upon a given hypotenuse, a right triangle equal to a given triangle.

53. To construct upon a given hypotenuse, a right triangle equal to a given square.

54. To construct (*a*) a triangle which shall have a given base, a given adjoining angle, and be equal to a given triangle; (*b*) a triangle equal to a given square; (*c*) a triangle equal to a given polygon.

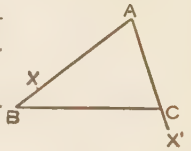
55. To construct (*a*) a parallelogram which shall have a given base, a given adjoining angle, and be equal to a given parallelogram; (*b*) a parallelogram equal to a given triangle; (*c*) a parallelogram equal to a given polygon.

56. To construct a line, DE, from D, a given point in AB of triangle ABC, so that DE bisects the triangle.



Analysis: After DE is drawn, $\triangle ABC = 2 \triangle ADE$ (Hyp.). But $\triangle ABC : \triangle ADE = AB \cdot AC : AD \cdot AE$ (?). Hence, $AB \cdot AC = 2\,(AD \cdot AE)$ (Ax. 6). $\therefore 2\,AD : AB = AC : x$ (?). Thus x (that is, AE) is a fourth proportional to three given lines.

57. To draw a line meeting two sides of a triangle and forming an isosceles triangle equal to the given triangle.



Analysis: Suppose AX is a leg of the required isosceles $\triangle$. $\therefore \triangle ABC : \triangle AXX' = AB \cdot AC : AX \cdot AX'$. But the $\measuredangle$ are equal and $AX = AX'$ (Hyp.).

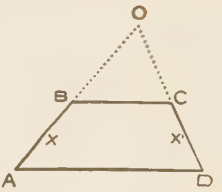
Hence, $AB \cdot AC = \overline{AX}^2$. $\therefore AX$ is a mean proportional between AB and AC.

58. To draw a line parallel to the base of a triangle which shall bisect the triangle.

59. To draw a line meeting two sides of a triangle forming an isosceles triangle equal to half the given triangle.

60. To draw a line parallel to the base of a triangle forming a triangle equal to one third of the original triangle.

61. To draw a line parallel to the base of a trapezoid so that the area is bisected.



Analysis: $\triangle OXX' = \frac{1}{2}\,(\triangle OAD + \triangle OBC)$ and is similar to $\triangle OBC$.

62. To construct two lines parallel to the base of a triangle, that shall trisect the area of the triangle.

63. To construct a triangle, having given its angles and its area.

Analysis: The required $\triangle$ is similar to any $\triangle$ containing the given $\measuredangle$. The given area may be a square. This reduces the problem to 401.

64. To find two straight lines in the ratio of two given polygons.

BOOK V

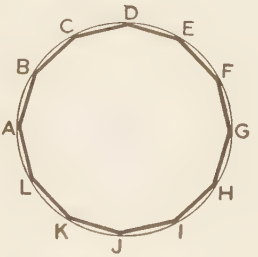
REGULAR POLYGONS. CIRCLES

THEOREMS AND DEMONSTRATIONS

402. A **regular polygon** is a polygon which is equilateral and equiangular.

PROPOSITION I. THEOREM

403. An equilateral polygon inscribed in a circle is regular.



Given: AG, an equilateral inscribed polygon.

To Prove: AG is regular.

Proof: chord AB = chord BC = chord CD = etc. (Hyp.).

$\therefore$ arc AB = arc BC = arc CD = etc. (196).

$\therefore$ arc AC = arc BD = arc CE = etc. (Ax. 3).

$\therefore \angle ABC = \angle BCD = \angle CDE =$ etc. (239).

That is, the polygon is equiangular.

$\therefore$ the polygon is regular (402). Q. E. D.

404. COROLLARY. **If the circumference of a circle is divided into any number of equal arcs, and the chords of these arcs are drawn, they form an inscribed regular polygon.**

Given: Arc AB = arc BC = arc CD = etc. and chords AB, etc.

To Prove: Polygon AG is a regular polygon.

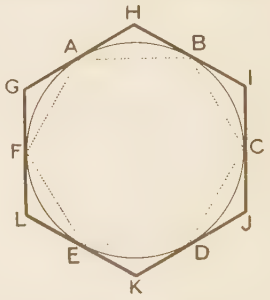
Proof: Chords AB, BC, CD, etc. are all equal. (197).

$\therefore$ the polygon is regular (403). Q. E. D.

405. COROLLARY. **If chords are drawn joining the alternate vertices of an inscribed regular polygon (having an even number of sides), another inscribed regular polygon is formed.**

Proposition II. Theorem

406. **If the circumference of a circle is divided into any number of equal parts, and tangents are drawn, at the several points of division, they form a circumscribed regular polygon.**



Given: Arcs AB, BC, CD, etc. all equal; and GH, HI, IJ, etc. tangents at A, B, C, etc.

To Prove: Polygon HK is regular.

Proof: Draw chords AB, BC, CD, etc.

In $\triangle ABH$, BCI, CDJ, etc. $AB = BC = CD$, etc. (197).

$\angle HAB = \angle HBA = \angle IBC = \angle ICB = \angle JCD$, etc. (237).

$\therefore$ these $\triangle$ are congruent and isosceles (76; 114).

$\therefore \angle H = \angle I = \angle J =$ etc. (27).

That is, polygon HK is equiangular.

Also $AH = HB = BI = IC = CJ$, etc. (24 or 206; 27).

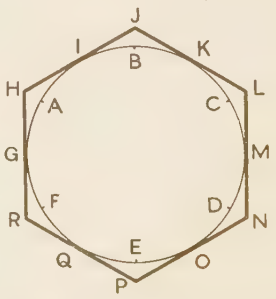
$\therefore HI = IJ = JK =$ etc. (Ax. 3).

That is, polygon HK is equilateral.

$\therefore$ polygon HK is regular (402)

Q.E.D.

407. Corollary. **If the circumference of a circle is divided into any number of equal parts and tangents are drawn at their midpoints, they form a circumscribed regular polygon.**



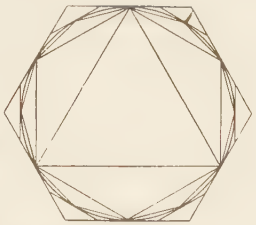
Given: (?) **To Prove:** (?).

Proof: Arcs AB, BC, CD, etc. are all equal (Hyp.).

Also arcs AI, IB, BK, KC, CM, etc. are all equal (Ax. 3).

$\therefore$ arcs IK, KM, MO, etc. are all equal (Ax. 3).

Therefore the polygon is regular (406). Q.E.D.



408. COROLLARY. If the vertices of an inscribed regular polygon are joined to the midpoints of the arcs subtended by the sides, another inscribed regular polygon is formed (having double the number of sides).

409. COROLLARY. If tangents are drawn at the midpoints of the arcs between adjacent points of contact of the sides of a circumscribed regular polygon, another circumscribed regular polygon is formed having double the number of sides. (?).

410. COROLLARY. The perimeter of an inscribed regular polygon is less than the perimeter of an inscribed regular polygon having twice as many sides, and the perimeter of a circumscribed regular polygon is greater than the perimeter of a circumscribed regular polygon having twice as many sides. (Ax. 12.)

Ex. 1. If in a regular dodecagon (Fig. of 403) all the diagonals are drawn from vertex A, how many degrees are there between adjacent pairs of diagonals, at A?

Ex. 2. In the figure of 406, how many degrees are there in each of the three angles at A? Prove in three ways that there are $120°$ in $\angle G$. If radii were drawn to the vertices of the inscribed polygon, what kind of triangles would be formed?

Ex. 3. Write a formula for finding the number of degrees in each angle of a regular polygon.

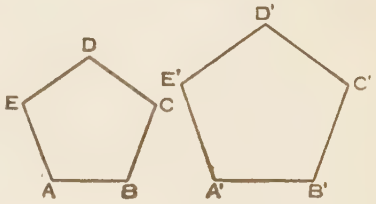
Ex. 4. Are the following figures regular polygons: a square? an isosceles triangle? a rectangle? a rhombus? an equilateral triangle?

Ex. 5. In the figure of 406, prove that a line from the center of the circle OI is the perpendicular bisector of chord BC. Then prove triangles OBI and OBM similar (M being midpoint of BC). Hence prove $OI : OB = OB : OM$.

Ex. 6. Can you explain how to divide a circle into three equal parts? into four equal parts?

Proposition III. Theorem

411. Two regular polygons having the same number of sides are similar.



Given: Regular n-gons AD and $A'D'$.

To Prove: They are similar.

Proof:

$$\angle A = \frac{(n-2)\,180^\circ}{n} \qquad (155).$$

$$\angle A' = \frac{(n-2)\,180^\circ}{n} \qquad (?).$$

$$\therefore \angle A = \angle A' \qquad (\text{Ax. } 1).$$

Similarly, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

That is, these polygons are mutually equiangular.

Now $AB = BC = CD = \text{etc.}$; $A'B' = B'C' = C'D' = \text{etc.}$ (402).

$$\therefore AB : A'B' = BC : B'C' = CD : C'D' = \text{etc.} \qquad (\text{Ax. } 3).$$

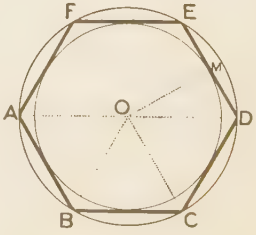
That is, the homologous sides are proportional.

Therefore the polygons are similar (301).

Q.E.D.

Proposition IV. Theorem

412. A circle can be circumscribed about, and a circle can be inscribed in, any regular polygon.



Given: Regular polygon $ABCDEF$.

To Prove: I. A circle can be circumscribed about the polygon.

II. A circle can be inscribed in the polygon.

Proof: I. Through three consecutive vertices, A, B, and C, describe a circumference, whose center is O. Draw radii OA, OB, OC, and draw line OD.

In $\triangle AOB$ and COD,	$AB = CD$	(402),
	$BO = CO$	(187).
Also	$\angle ABC = \angle BCD$	(402),
	$\angle OBC = \angle OCB$	(55).
Subtracting,	$\angle ABO = \angle OCD$	(Ax. 2).
	$\therefore$ $\triangle AOB$ is congruent to $\triangle COD$	(52).
	$\therefore$ $AO = OD$	(27).

Hence the arc passes through D, and in like manner it may be proved that it passes through E and F.

That is, a circle can be circumscribed about the polygon.

II. AB, BC, CD, DE, etc. are equal chords (402).

$\therefore$ they are equally distant from the center (208).

That is, a circle described, using O as a center and OM as a radius, will touch every side of the polygon. (202).

Hence a circle can be inscribed (221). Q.E.D.

413. The **radius** of a regular polygon is the radius of the circumscribed circle. The radius of the inscribed circle is called the **apothem.** The **center** of a regular polygon is the common center of the circumscribed and inscribed circles.

414. The **central angle** of a regular polygon is the angle included between two radii drawn to the ends of a side.

415. THEOREM. **Each central angle of a regular n-gon $= \frac{360^\circ}{n}$.**

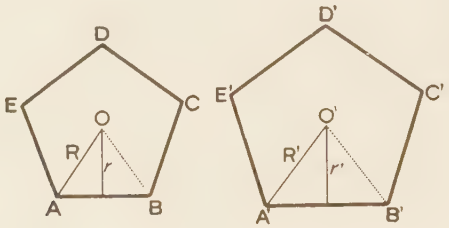
416. COROLLARY. **Each exterior angle of a regular n-gon $= \frac{360^\circ}{n}$.**

417. COROLLARY. **The radius drawn to any vertex of a regular polygon bisects the angle at the vertex** (95).

418. COROLLARY. **The central angles of regular polygons having the same number of sides are equal.**

Proposition V. Theorem

419. The perimeters of two regular polygons having the same number of sides are to each other as their radii and also as their apothems.



Given: Regular n-gons, EC, with perimeter P, radius R, apothem r; and $E'C'$ with perimeter P', radius R', apothem r'.

To Prove: $P : P' = R : R' = r : r'$.

Proof: Draw radii OB and $O'B'$.

In $\triangle AOB$ and $A'O'B'$, $\angle AOB = \angle A'O'B'$ (418).

$AO = BO$ (187).

$A'O' = B'O'$ (?).

$\therefore \frac{AO}{A'O'} = \frac{BO}{B'O'}$ (Ax. 3).

$\therefore \triangle AOB$ is similar to $\triangle A'O'B'$ (306).

$\therefore \frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$ (313).

But these polygons are similar (411).

$\therefore \frac{P}{P'} = \frac{AB}{A'B'}$ (317).

$\therefore \frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}$ (Ax. 1).

Q.E.D.

Proposition VI. Theorem

420. **The areas of two regular polygons having the same number of sides are to each other as the squares of their radii and also as the squares of their apothems.**

Given: Regular n-gons, EC whose area is K, radius R, apothem r; and $E'C'$ whose area is K', radius R', apothem r'.

To Prove: $K : K' = R^2 : R'^2 = r^2 : r'^2$.

Proof: As in 419, $\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$.

Then $\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$ (287).

But these polygons are similar (411).

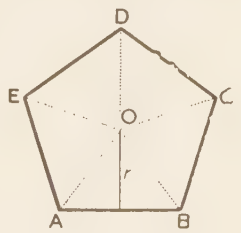
$\therefore \frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$ (376).

$\therefore \frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$ (Ax. 1).

Q.E.D.

Proposition VII. Theorem

421. **The area of a regular polygon is equal to half the product of the perimeter by the apothem.**



Given: (?).

To Prove: (?).

Proof: Draw radii to all the vertices, forming several isosceles triangles.

Area of $\triangle AOB = \frac{1}{2} AB \cdot r$
Area of $\triangle BOC = \frac{1}{2} BC \cdot r$
Area of $\triangle COD = \frac{1}{2} CD \cdot r$
etc., etc. (364).

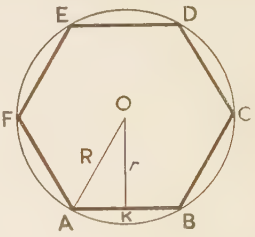
Area of polygon $= \frac{1}{2}(AB + BC + CD + \text{etc.}) \cdot r$ (Ax. 2).

Substituting, area $= \frac{1}{2} P \cdot r$ (Ax. 6).

Q.E.D.

Proposition VIII. Theorem

422. If the number of sides of an inscribed regular polygon is increased indefinitely, the apothem approaches the radius as a limit.



Given: Polygon FC inscribed in $\odot O$; apothem $= r$; radius $= R$.

To Prove: That as the number of sides is indefinitely increased, r approaches R as a limit.

Proof: In the $\triangle AOK$,

$$R < r + AK \qquad \text{(Ax. 12)}.$$

Or

$$R - r < AK \qquad \text{(Ax. 7)}.$$

Now, as the number of sides of the polygon is indefinitely increased, AB is indefinitely decreased.

Hence $\frac{1}{2}\,AB$, or AK, approaches zero as a limit.

$\therefore$ $R - r$ approaches zero (because $R - r < AK$).

That is, r approaches R as a limit (227). Q.E.D.

Note. It is evident that if the difference between two variables approaches zero, either

(1) one is approaching the other as a limit; or

(2) both are approaching some third quantity as their limit.

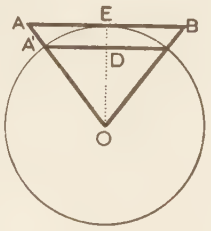
423. Theorem. The circumference of a circle is less than the perimeter of any circumscribed polygon.

By drawing tangents at the midpoints of the included arcs of a circumscribed polygon, another circumscribed polygon is formed; the perimeter of this polygon is less than the perimeter of the given polygon. This can be continued indefinitely, decreasing the perimeter of the polygons. Hence there can be no circumscribed polygon whose perimeter can be the least of all such polygons; because, by increasing the number of sides, the perimeter is lessened. Hence the circumference must be less than the perimeter of any circumscribed polygon.

424. Theorem. If the number of sides of an inscribed regular polygon and of a circumscribed regular polygon is indefinitely increased.

I. The perimeter of each polygon approaches the circumference of the circle as a limit.

II. The area of each polygon approaches the area of the circle as a limit.



Given: A circle O, whose circumference is C and whose area is S; AB and $A'B'$, sides of regular circumscribed and inscribed polygons, having the same number of sides; P and P', their perimeters; K and K', their areas.

To Prove: That if the number of sides is indefinitely increased:

I. P approaches C and P' approaches C as limit.

II. K approaches S and K' approaches S as limit.

Proof: I. The polygons are similar (411).

$$\therefore \frac{P}{P'} = \frac{OE}{OD} \qquad (419).$$

Now, if the number of sides of these polygons is indefinitely increased, OD approaches OE (422).

Hence $\frac{OE}{OD}$ approaches 1. That is, $\frac{P}{P'}$ approaches 1, or P and P' approach equality; that is, they approach the same constant as a limit.

But $P > C$ and $C > P'$ and C is constant.

Hence P approaches C and P' approaches C. Q.E.D.

II. $$\frac{K}{K'} = \frac{\overline{OE}^2}{\overline{OD}^2} \qquad (420).$$

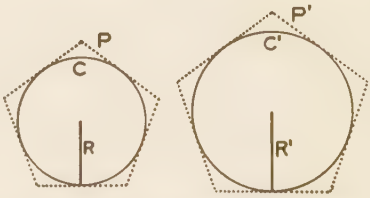
If the number of sides of these polygons is indefinitely increased, $\overline{OD}^2$ approaches $\overline{OE}^2$, and thus $\frac{\overline{OE}^2}{\overline{OD}^2}$ approaches unity.

(The argument continues the same as in I.)

NOTE. The theorems of 423 and 424 are considered so evident, and rigorous proofs (as in the case of the demonstrations for many fundamental principles in mathematics) are so difficult for young students to comprehend, that it is advisable to omit the profound demonstrations and insert only simple explanations.

PROPOSITION IX. THEOREM

425. The circumferences of two circles are to each other as their radii.



Given: Two ⓢ whose radii are R and R' and circumferences, C and C' respectively.

To Prove: $C : C' = R : R'$.

Proof: Circumscribe regular polygons (having the same number of sides) about these ⓢ and let P and P' denote their perimeters. Then $P : P' = R : R'$ (419).

Hence $P \cdot R' = P' \cdot R$ (?).

Now suppose the number of sides of these polygons to be indefinitely increased,

P approaches C (424).

P' approaches C' (?).

$\therefore P \cdot R'$ approaches $C \cdot R'$.

Also $P' \cdot R$ approaches $C' \cdot R$.

Hence $C \cdot R' = C' \cdot R$ (229).

Therefore $C : C' = R : R'$ (281).

Q.E.D.

426. THEOREM. **The ratio of any circumference to its diameter is constant for all circles. That is, any circumference divided by its diameter is the same as any other circumference divided by its diameter.**

Proof: $\dfrac{C}{C'} = \dfrac{R}{R'}$ (425).

But $\dfrac{R}{R'} = \dfrac{\frac{1}{2}D}{\frac{1}{2}D'} = \dfrac{D}{D'}$ (Ax. 6).

$$\therefore \frac{C}{C'} = \frac{D}{D'} \qquad \text{(Ax. 1).}$$

$$\therefore \frac{C}{D} = \frac{C'}{D'} \qquad \text{(282).}$$

That is, $\dfrac{\text{circumference}}{\text{diameter}}$ = a constant for all ⊙. Q.E.D.

427. Definition of π (pi). The constant ratio of a circumference to its diameter is called π. That is, $\dfrac{C}{D} = \pi$.

The numerical value of π is 3.141592, or $3\frac{1}{7}$, approximately. (This is computed in 453.)

428. Formula. Let C = the circumference of a circle with radius R. Then $\dfrac{C}{2R} = \pi$. (427).

$$\therefore C = 2\pi R \qquad \text{(Ax. 3).}$$

Ex. 1. Find the circumference of a circle the radius of which is 12 in.

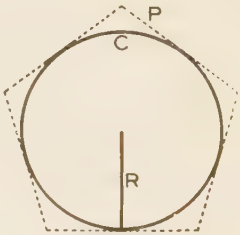
Ex. 2. Find the radius of a circle the circumference of which is 66 feet.

Historical Note. Gottfried Wilhelm von Leibnitz, a German philosopher, mathematician, and man of affairs, was born in 1646 and died in 1716. He could read Latin easily at 12, and wrote some Latin verse. While at the University of Leipsic he became acquainted with Francis Bacon, Kepler, Galileo, and Descartes, modern thinkers who had revolutionized science and philosophy. He resolved to study mathematics, but not until he had reached his majority did he throw himself into deep mathematical research. It was while he was living in Paris and Mainz that he announced his imposing discoveries in natural philosophy, mathematics, mechanics, optics, hydrostatics, pneumatics, and nautical science. In mathematics, he was the discoverer of the differential and integral calculus. He possessed a marvelous ability for rapid and continuous work. Even in traveling his time was employed in solving mathematical problems. He is described as moderate in habits, quick of temper, charitable in judgment of others, tolerant of differences of opinion, but impatient of contradiction on small matters and desirous of honor.

Leibnitz

Proposition X. Theorem

429. The area of a circle is equal to half the product of its circumference by its radius.



Given: $\odot$ with circumference C, area S, radius R.

To Prove: $S = \frac{1}{2} C \cdot R$.

Proof: Circumscribe a regular polygon about the circle; denote its area by K and perimeter by P.

Now $K = \frac{1}{2} P \cdot R$ (421).

Suppose the number of sides of the polygon is indefinitely increased. K approaches S, and P approaches C (424).

$\frac{1}{2} P \cdot R$ approaches $\frac{1}{2} C \cdot R$ as a limit.

Hence $S = \frac{1}{2} C \cdot R$ (229). Q.E.D.

430. Formula. Let S = the area of a circle whose circumference = C, and whose radius = R.

Then $S = \frac{1}{2} C \cdot R$ (429).

But $C = 2\pi R$ (428).

Substituting, $\boldsymbol{S = \pi R^2}$ (Ax. 6).

Ex. 1. Find the area of a circle the radius of which is 8 in.

Ex. 2. Find the radius of a circle the area of which is 500 sq. ft.

431. Corollary. **The areas of two circles are to each other as the squares of their radii, and as the squares of their diameters.**

To Prove: $S : S' = R^2 : R'^2 = D^2 : D'^2$.

Proof: $S = \pi R^2$, and $S' = \pi R'^2$ (430).

Dividing, $\frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2}$ (Ax. 3).

Now $\frac{S}{S'} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2} D)^2}{(\frac{1}{2} D')^2} = \frac{\frac{1}{4} D^2}{\frac{1}{4} D'^2} = \frac{D^2}{D'^2}$ (Ax. 6). Q.E.D.

432. COROLLARY. **The area of a sector is the same part of the circle as its central angle is of 360°.** (Ax. 1.)

433. FORMULA. An arc : circum. = central $\angle$: 360° (231).

$$\therefore \textbf{arc} : 2\pi R = \angle : 360°.$$

NOTE. If any two of the three quantities, arc, R, $\angle$, are known, the remaining one can be found by this proportion.

434. FORMULA. Sector : area of $\odot$ = central $\angle$: 360° (432).

$$\therefore \textbf{sector} : \pi R^2 = \angle : 360°.$$

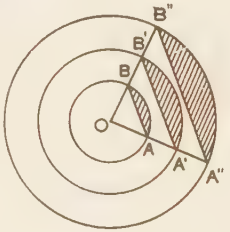
NOTE. If any two of the three quantities, sector, R, $\angle$, are known, the remaining one can be found by this proportion.

435. FORMULA. Sector : area of $\odot$ = arc : circum. (Ax. 1).

$$\therefore \text{sector} : \pi R^2 = \text{arc} : 2\pi R \quad \text{(Ax. 6)}.$$

$$\therefore \textbf{sector} = \tfrac{1}{2} R \cdot \textbf{arc} \quad \textbf{(280)}.$$

436. **Similar arcs, similar sectors,** and **similar segments** are those which correspond to equal central angles, in unequal circles.



Thus, AB, $A'B'$, $A''B''$ are similar arcs; AOB, $A'OB'$, and $A''OB''$ are similar sectors; and the shaded segments are similar segments.

437. THEOREM. **Similar arcs are to each other as their radii.**

Given: Arcs whose lengths are a and a', radii R and R'.

To Prove: $a : a' = R : R'$.

Proof:

$$\frac{a}{2\pi R} = \frac{\angle}{360°}, \text{ and } \frac{a'}{2\pi R'} = \frac{\angle}{360°} \quad \text{(433)}.$$

$$\therefore \frac{a}{2\pi R} = \frac{a'}{2\pi R'} \quad \text{(Ax. 1)}.$$

$$\therefore \frac{a}{a'} = \frac{2\pi R}{2\pi R'} = \frac{R}{R'} \quad \text{(282)}.$$

Q.E.D.

438. Theorem. Similar sectors are to each other as the squares of their radii.

Given: Sectors whose areas are T and T', radii R and R'.

To Prove: $T : T' = R^2 : R'^2$.

Proof:

$$\frac{T}{\pi R^2} = \frac{\angle}{360}, \text{ and } \frac{T'}{\pi R'^2} = \frac{\angle}{360} \quad (434).$$

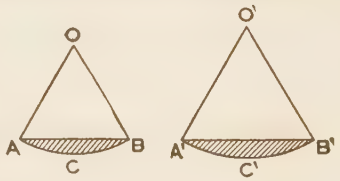
$$\therefore \frac{T}{\pi R^2} = \frac{T'}{\pi R'^2} \quad (\text{Ax. } 1).$$

$$\therefore \frac{T}{T'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2} \quad (282).$$

Q.E.D.

Proposition XI. Theorem

439. Similar segments are to each other as the squares of their radii.



Given: Similar segments ABC and $A'B'C'$.

To Prove: Segment ABC : segment $A'B'C' = R^2 : R'^2$.

Proof: $\triangle AOB$ and $A'O'B'$ are similar (306).

$$\therefore \frac{\triangle AOB}{\triangle A'O'B'} = \frac{R^2}{R'^2} \quad (375).$$

Also $$\frac{\text{sector } OACB}{\text{sector } O'A'C'B'} = \frac{R^2}{R'^2} \quad (438).$$

$$\therefore \frac{\text{sector } OACB}{\text{sector } O'A'C'B'} = \frac{\triangle AOB}{\triangle A'O'B'} \quad (\text{Ax. } 1).$$

$$\therefore \frac{\text{sector } OACB}{\triangle AOB} = \frac{\text{sector } O'A'C'B'}{\triangle A'O'B'} \quad (282).$$

$$\therefore \frac{\text{sector } OACB - \triangle AOB}{\triangle AOB} = \frac{\text{sector } O'A'C'B' - \triangle A'O'B'}{\triangle A'O'B'} \quad (285).$$

That is, $$\frac{\text{segment } ABC}{\triangle AOB} = \frac{\text{segment } A'B'C'}{\triangle A'O'B'} \quad (\text{Ax. } 6).$$

$$\therefore \frac{\text{segment } ABC}{\text{segment } A'B'C'} = \frac{\triangle AOB}{\triangle A'O'B'} = \frac{R^2}{R'^2} \quad (282).$$

Q.E.D.

ORIGINAL EXERCISES (THEOREMS)

1. The central angle of a regular polygon is the supplement of the angle of the polygon.

2. An equiangular polygon inscribed in a circle is regular (if the number of its sides is odd).

3. An equiangular polygon circumscribed about a circle is regular. [Draw radii and apothems.]

4. The sides of a circumscribed regular polygon are bisected at the points of contact.

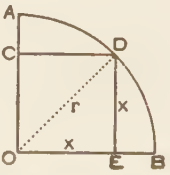
5. The diagonals of a regular pentagon are equal.

6. The diagonals drawn from any vertex of a regular n-gon divide the angle at that vertex into $n - 2$ equal parts.

7. If a regular polygon is inscribed in a circle, and another regular polygon having the same number of sides is circumscribed about it, the radius of the circle is a mean proportional between the apothem of the inner and the radius of the outer polygon.

8. The area of the square inscribed in a sector the central angle of which is a right angle, is equal to half the square of the radius.

[Find x^2, the area of $OEDC$.]

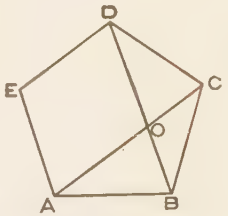


9. The apothem of an equilateral triangle is one third the altitude of the triangle.

10. The chord that bisects a radius of a circle at right angles is the side of the inscribed equilateral triangle.

[Prove that the central $\angle$ subtended is 120°.]

11. If $ABCDE$ is a regular pentagon, and diagonals AC and BD are drawn, meeting at O:



(a) $AO = AB$.
(b) AO is $\parallel$ to ED.
(c) $\triangle BOC$ is similar to $\triangle BDC$.
(d) $\angle ACB = 36°$.
(e) AC is divided into mean and extreme ratio at O.

12. The altitude of an equilateral triangle is three fourths the diameter of the circumscribed circle.

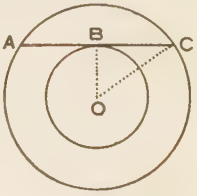
13. The apothem of an inscribed regular hexagon equals half the side of an inscribed equilateral triangle.

14. The area of a circle is four times the area of another circle described upon its radius as a diameter.

15. The area of an inscribed square is half the area of the circumscribed square.

16. An equilateral polygon circumscribed about a circle is regular (if the number of its sides is odd).

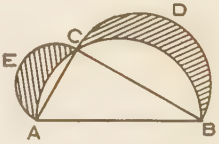
17. The sum of the circles described upon the legs of a right triangle as diameters is equal to the circle described upon the hypotenuse as a diameter.



18. A circular ring (the area between two concentric circles) is equal to the circle described upon the chord of the larger circle, which is tangent to the less, as a diameter.

Proof: Draw radii OB, OC. $\triangle OBC$ is rt. $\triangle$ (?); and $\overline{OC}^2 - \overline{OB}^2 = \overline{BC}^2$ (?). Etc.

19. If semicircles are described upon the three sides of a right triangle (on the same side of the hypotenuse), the sum of the two crescents thus formed is equal to the area of the triangle.

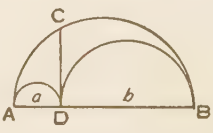


Proof: $\begin{cases} \text{Entire figure} = \frac{1}{8}\pi\overline{AB}^2 + \text{crescent } BDC + \text{crescent } AEC \text{ (?)} \\ \text{Entire figure} = \frac{1}{8}\pi\overline{AC}^2 + \frac{1}{8}\pi\overline{BC}^2 + \triangle ABC \text{ (?)} \end{cases}$

Now use Ax. 1; etc.

20. Show that the theorem of Ex. 19 is true in the case of a right triangle whose legs are 18 and 24.

21. If from any point in a semicircle a line is drawn perpendicular to the diameter and if semicircles are described on the two segments of the original diameter as diameters, the area of the surface bounded by these three semicircles equals the area of a circle whose diameter is the perpendicular first drawn.



Proof: Area $= \frac{1}{2}\pi\left(\frac{a+b}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{b}{2}\right)^2 =$ etc.

22. Show that the theorem of Ex. 21 is true in the case of a circle with diameter AB equal to 25 and AD equal to 5.

23. If the sides of a circumscribed regular polygon are tangent to the circle at the vertices of an inscribed regular polygon, each vertex of the outer lies on the prolongation of the apothems of the inner polygon, drawn perpendicular to the several sides.

24. The sum of the perpendiculars drawn from any point within a regular n-gon to the several sides is constant [$= n \cdot$ apothem].

25. The area of a circumscribed equilateral triangle is four times the area of the inscribed equilateral triangle.

26. If a point is taken dividing the diameter of a circle into two parts and circles are described upon these parts as diameters, the sum of the circumferences of these two circles equals the circumference of the original circle.

27. Show that the theorem of Ex. 26 is true in the case of a circle the segments of the diameter of which are 7 and 12.

28. The area of an inscribed regular octagon is equal to the product of the diameter by the side of the inscribed square.

29. If squares are described on the six sides of a regular hexagon (externally), the twelve exterior vertices of these squares are the vertices of a regular 12-gon.

30. If the alternate vertices of a regular hexagon are joined by drawing diagonals, another regular hexagon is formed. Also its area is one third of the original hexagon.

31. Show that the theorem of Ex. 18 is true in the case of two concentric circles whose radii are 34 and 16.

32. In the same or equal circles two sectors are to each other as their central angles.

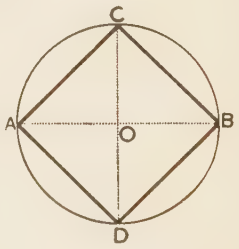
33. If the diameter of a circle is 10 in. and a point is taken dividing the diameter into segments with lengths 4 in. and 6 in., and on these segments as diameters semicircles are described on opposite sides of the diameter, these arcs form a curved line which divides the original circle into two parts in the ratio of 2 : 3.

34. If the diameter of a circle is d and a point is taken dividing the diameter into segments with lengths a and $d - a$, and on these segments as diameters semicircles are described on opposite sides of the diameter, these arcs form a curved line which divides the original circle into two parts in the ratio of $a : d - a$.

CONSTRUCTION PROBLEMS

Proposition XII. Problem

440. To inscribe a square in a given circle.



Given: The circle O.

Required: To inscribe a square.

Construction: Draw any diameter, AB, and another diameter, CD, $\perp$ to AB. Draw AC, BC, BD, AD.

Statement: $ACBD$ is an inscribed square. Q.E.F.

Proof: The central $\measuredangle$ are all equal (42).

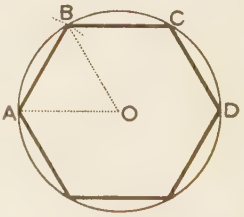
$\therefore$ arcs AC, CB, BD, DA are equal (193).

$\therefore$ $ACBD$ is an inscribed square (404). Q.E.D.

Proposition XIII. Problem

441. To inscribe a regular hexagon in a given circle.

Given: (?). **Required:** (?).



Construction: Draw any radius, AO. At A, with radius = to AO, describe arc intersecting the given $\odot$ at B. Draw AB.

Statement: AB is the side of an inscribed regular hexagon.

Proof: Draw BO. $\triangle ABO$ is equilateral (Const.).

$\therefore$ $\triangle ABO$ is equiangular (56).

$\therefore$ $\angle AOB = 60°$ (109).

That is, arc $AB = \frac{1}{6}$ of the circumference; and if arc AB is used as a unit, it divides the circumference into 6 equal arcs. If the chords are drawn, an inscribed regular hexagon is formed (404).

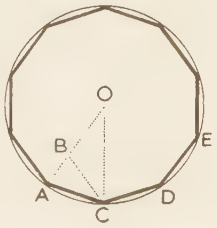
Q.E.D.

Proposition XIV. Problem

442. To inscribe a regular decagon in a given circle.

Given: (?).

Required: (?).



Construction: Draw any radius AO. Divide it into mean and extreme ratio (by 349), having the larger segment next to the center. Taking A as a center and OB as a radius, draw an arc cutting $\odot$ at C. Draw AC, BC, OC.

Statement: AC is a side of the inscribed regular decagon. Q.E.F.

Proof:

$AO : BO = BO : AB$ (Const.).

Substituting, $AO : AC = AC : AB$ (Ax. 6).

$\therefore$ $\triangle ABC$ and AOC are similar (306).

$\therefore$ First, $\angle ACB = \angle O$ (312).

Second, $\triangle ABC$ is isosceles (similar to $\triangle AOC$).

$\therefore AC = BC$ (24).

But $AC = BO$ (Const.).

$\therefore BC = BO$ (Ax. 1).

$\therefore \angle BCO = \angle O$ (55).

Now $\angle ACO = 2\angle O$ (Ax. 4).

$\therefore \angle A = 2\angle O$ (55).

And $\angle O = 1\angle O$

Adding, the $\measuredangle$ of $\triangle ACO = 5\angle O$ (Ax. 2).

$\therefore 5\angle O = 180°$ (104).

$\therefore \angle O = 36° = \frac{1}{10}$ of $360°$ (Ax. 3).

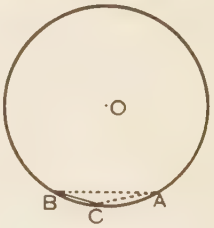
$\therefore$ arc $AC = \frac{1}{10}$ of the circumference (193).

That is, if arc AC is used as a unit it divides the circumference into ten equal arcs; and if the chords are drawn, an inscribed regular decagon is formed. (404).

Q.E.D.

Proposition XV. Problem

443. To inscribe a regular 15-gon (pentadecagon) in a given circle.



Given: (?). **Required:** (?).

Construction: Draw AB, the side of an inscribed hexagon, and AC, the side of an inscribed decagon. Draw BC.

Statement: BC is the side of an inscribed regular 15-gon. Q.E.F.

Proof: Arc BC = arc AB − arc AC
$= \frac{1}{6} - \frac{1}{10}$ of circumference (Const.).
$= \frac{1}{15}$ of circumference.

That is, if arc BC is used as a unit, it divides the circumference into fifteen equal arcs; and if the chords are drawn, an inscribed regular pentadecagon is formed (404). Q.E.D.

444. To inscribe in a given circle:

I. A regular 8-gon, a regular 16-gon, a regular 32-gon, etc.

II. A regular 12-gon, 24-gon, etc.

III. A regular 30-gon, 60-gon, etc.

Construction: I. Inscribe a square. (440.)
Bisect the arcs and draw the chords. **Proof:** (404).

II. Inscribe a regular hexagon, etc.

III. Inscribe a regular pentadecagon, etc.

445. To inscribe an equilateral triangle in a circle.

Construction: Join the alternate vertices of an inscribed regular hexagon. **Proof:** (?) (405).

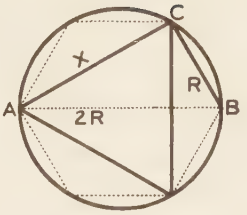
446. To inscribe a regular pentagon in a given circle.

447. To circumscribe a regular polygon about a circle.

Construction: Inscribe a polygon having the same number of sides. At the several vertices draw tangents.

Statement: (?). **Proof:** (?) (406).

FORMULAS



448. Sides of inscribed polygons.

1. The side of inscribed equilateral triangle $= R\sqrt{3}$.

Proof:

$\angle ACB$ is a rt. $\angle$ (240).

$AB = 2R$ (189).

$BC = R$ (441).

$x^2 = (2R)^2 - R^2$ (335).

$x = R\sqrt{3}$. Q.E.D.

2. The side of an inscribed square $= R\sqrt{2}$.

Proof: In fig. of 440

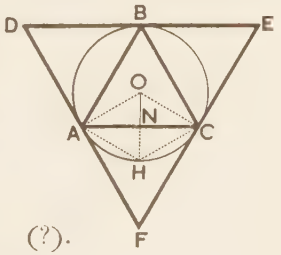
$AO = OC = R$

$\overline{AC}^2 = R^2 + R^2$ (334).

$\therefore AC = R\sqrt{2}$. Q.E.D.

3. The side of an inscribed regular hexagon $= R$ (441).

4. The side of an inscribed regular decagon $= \frac{1}{2}R(\sqrt{5} - 1)$ (352; 442).



449. Sides of circumscribed polygons.

1. The side of a circumscribed equilateral $\triangle = 2R\sqrt{3}$.

Proof: $\angle DAB = \angle DBA = \angle D = 60°$ (?).

$\therefore \triangle ABD$ is equilateral.

$AD = AB = R\sqrt{3}$ (448).

$\therefore DF = 2R\sqrt{3}$. Q.E.D.

2. The side of a circumscribed square $= 2R$ (?).

3. **The side of a circumscribed regular hexagon** $= \frac{2}{3} R\sqrt{3}$.

Proof: This side = a side of an equilateral $\triangle$ with altitude R. Let x = this side.

$$\therefore x^2 - (\tfrac{1}{2}x)^2 = R^2 \ (?) \quad \therefore x = \tfrac{2}{3} R\sqrt{3}.$$

450. In an equilateral triangle, **apothem** $= \frac{1}{2} R$.

Proof: Fig. of 449. Bisect arc AC at H.

Draw OA, OC, AH and CH.

Figure $AOCH$ is a rhombus (448, 3).

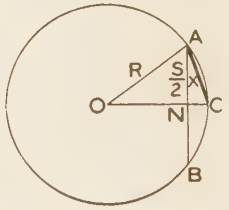
$\therefore ON = \frac{1}{2} OH = \frac{1}{2} R$ (135).

Q.E.D.

Proposition XVI. Problem

451. In a circle whose radius is R is inscribed a regular polygon whose side is s; to find the formula for the side of an inscribed regular polygon having double the number of sides.

Given: $AB = s$, a side of an inscribed regular polygon in $\odot$ whose radius is R; C, the midpoint of arc AB; chord AC.



Required: To find the value of AC, the side of a regular polygon having double the number of sides and inscribed in the same circle.

Construction: Draw radii OA and OC.

Computation: OC bisects AB at right $\measuredangle$ (83).

In rt. $\triangle AON$, O is an acute $\angle$. Hence in $\triangle AOC$,

$$\overline{AC}^2 = \overline{OA}^2 + \overline{OC}^2 - 2 \cdot OC \cdot ON \quad (337).$$

But $AO = OC = R$ (187).

And $ON = \sqrt{R^2 - (\frac{1}{2}s)^2} = \frac{1}{2}\sqrt{4R^2 - s^2}$ (335).

Substituting, $\overline{AC}^2 = 2R^2 - 2R \cdot \frac{1}{2}\sqrt{4R^2 - s^2}$ (Ax. 6).

$$\therefore AC = \sqrt{2R^2 - R\sqrt{4R^2 - s^2}}.$$

Q.E.F.

452. Formula. If $R = 1$, and given side $= s$, the side of a regular polygon having twice as many sides $= \sqrt{2 - \sqrt{4 - s^2}}$.

Proposition XVII. Problem

453. To find the approximate numerical value of π.

Given: A circle whose diameter $= D$; circumference $= C$.

Required: The value of π, that is $\frac{C}{D}$.

Method: 1. We may select a $\odot$ of *any* diameter. (426.)

2. We can compute the side and the perimeter of some inscribed regular polygon. (448.)

3. We can compute the side and perimeter of another inscribed regular polygon having double the number of sides. (451.)

4. We can now compute the side and perimeter of a third inscribed regular polygon, having still double the number of sides. (451.)

5. By continuing this process until the consecutive perimeters differ very slightly, we can find the approximate value of the circumference.

6. Thus, knowing both C and D, we know $\frac{C}{D}$ or π.

Computation: 1. For simplicity, take $D = 2$, and $R = 1$.

2. We will select the regular hexagon as the first polygon.

$$\therefore s_6 = 1, \text{ and } P_6 = 6.$$

3. $\therefore s_{12} = \sqrt{2 - \sqrt{4 - {s_6}^2}} = \sqrt{2 - \sqrt{3}} = .5176381$ (452).

and $P_{12} = 6.2116572.$

4. Hence $s_{24} = \sqrt{2 - \sqrt{4 - {s_{12}}^2}} = \sqrt{2 - \sqrt{4 - (.5176381)^2}}$

$= .2610524.$

Also $P_{24} = 6.2652576.$

5. By continuing, $s_{3072} = .002045.$

Also $P_{3072} = 6.283184.$

6. It now appears that, approximately, $C = 6.283184.$

Hence $\pi = \frac{6.283184}{2} = 3.141592^{+}.$ Q.E.F.

This calculation is tabulated for reference.

$s_6 = 1$, $\therefore P_6 = 6$. $s_{192} = 0.032723$, $\therefore P_{192} = 6.282904$.
$s_{12} = 0.517638$, $\therefore P_{12} = 6.211657$. $s_{384} = 0.016362$, $\therefore P_{384} = 6.283115$.
$s_{24} = 0.261052$, $\therefore P_{24} = 6.265257$. $s_{768} = 0.008181$, $\therefore P_{768} = 6.283169$.
$s_{48} = 0.130806$, $\therefore P_{48} = 6.278700$. $s_{1536} = 0.004091$, $\therefore P_{1536} = 6.283180$.
$s_{96} = 0.065438$, $\therefore P_{96} = 6.282053$. $s_{3072} = 0.002045$, $\therefore P_{3072} = 6.283184$.

ORIGINAL EXERCISES (NUMERICAL)

Mensuration of Regular Polygons and the Circle

1. Find the angle and the central angle of :
(*i*) a regular pentagon ; (*ii*) a regular octagon ; (*iii*) a regular dodecagon ; (*iv*) a regular 20-gon.

2. Find the area of a regular hexagon whose side is 8.

3. Find the area of a regular hexagon whose apothem is 4.

4. In a circle whose radius is 10 are inscribed an equilateral triangle, a square, and a regular hexagon. Find the perimeter, the apothem, and the area of each.

5. About a circle whose radius is 10 are circumscribed an equilateral triangle, a square, and a regular hexagon. Find the perimeter and the area of each.

6. Find the circumference and the area of a circle whose radius is 5 inches. [Use $\pi = 3\frac{1}{7}$.]

7. Find the circumference and the area of a circle whose diameter is 42 centimeters.

8. The radius of a certain circle is 9 meters. What is the radius of a second circle whose circumference is twice as long as the first? of a third circle whose area is twice as great as the first?

9. If the circumference of a circle is 55 yards, what is its diameter?

10. If the area of a circle is $113\frac{1}{4}$ square meters, what is its radius?

11. In a circle whose radius is 35 there is a sector whose angle is 40°. Find the length of the arc and the area of the sector.

12. The area of a circle is $6\frac{1}{4}$ times the area of another. If the radius of the smaller circle is 12, what is the radius of the larger circle?

13. If the angle of a sector is 72° and its arc is 44 inches, what is the radius of the circle? What is the area of the sector?

14. In a circle whose radius is 7 find the area of the segment whose central angle is 120°. [Required area = $\frac{1}{3}$ (area of $\odot$ − area eq. $\triangle$.]

15. If the radius of a circle is 4 feet, what is the area of a segment whose arc is 60°? of a segment whose arc is a quadrant?

16. Find the area of a circle inscribed in a square whose area is 75.

17. Find the area of an equilateral triangle inscribed in a circle whose area is $441\,\pi$ square meters.

18. If the length of a quadrant is 8 inches, what is the radius?

19. Find the length of an arc subtended by the side of an inscribed regular 15-gon if the radius is $4\frac{2}{3}$ inches.

20. The side of an equilateral triangle is 10. Find the areas of its inscribed and circumscribed circles.

21. Find the perimeter and the area of a segment whose chord is the side of an inscribed regular hexagon, if the radius of a circle is $5\frac{1}{4}$.

22. A circular lake 9 rods in diameter is surrounded by a walk $\frac{1}{2}$ rod wide. What is the area of the walk?

23. A locomotive driving wheel is 7 feet in diameter. How many revolutions will it make in running a mile?

24. What is the number of degrees in the central angle whose arc is as long as the radius?

25. Find the side of the square equal to a circle whose diameter is 4.2 meters.

26. Find the radius of that circle equal to a square whose side is 5.5 inches.

27. Find the radius of the circle which divides a given circle whose radius is $10\frac{1}{2}$ into two equal parts.

28. Three equal circles are each tangent to the other two and the diameter of each is 40 feet. Find the area between these circles.

[Required area = area of an eq. $\triangle$ minus area of three sectors.]

29. Find the area of the three segments of a circle whose radius is $5\sqrt{3}$, formed by the sides of the inscribed equilateral triangle.

30. If a cistern can be emptied in 5 hours by a 2-inch pipe, how long will be required to empty it by a 1-inch pipe?

31. Find the side, the apothem, and the area of a regular decagon inscribed in a circle whose radius is 6 feet.

32. What is the area of the circle circumscribed about an equilateral triangle whose area is $48\sqrt{3}$?

33. The circumferences of two concentric circles are 40 inches and 50 inches. Find the area of the circular ring between them.

34. A circle has an area of 80 square feet. Find the length of an arc of 80°.

35. Find the angle of a sector whose perimeter equals the circumference.

36. Find the angle of a sector whose area is equal to the square of the radius.

37. Find the area of a regular octagon inscribed in a circle whose radius is 20.

[Inscribe square, then octagon. Draw radii of octagon. Find area of one isosceles $\triangle$ formed, whose altitude is half the side of the square.]

38. A rectangle whose length is double its width, a square, an equilateral triangle, and a circle all have the same perimeter, namely, 132 meters. Which has the greatest area? the least?

39. Through a point without a circle whose radius is 35 inches two tangents are drawn, forming an angle of 60°. Find the perimeter and the area of the figure bounded by the tangents and their smaller intercepted arc.

40. In a circle whose radius is 12 are two parallel chords which subtend arcs of 60° and 90° respectively. Find the perimeter and the area of the figure bounded by these chords and their intercepted arcs.

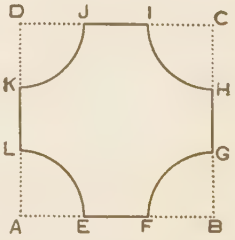
41. A quarter mile race track is to be laid out, having parallel sides but semicircular ends with radius 105 feet. Find the length of the parallel sides.

42. If the diameter of the earth is 7920 miles, how far at sea can the light from a lighthouse 150 feet high be seen?

43. The diameter of a circle is 18 inches. Find the area of the figure between this circle and the circumscribed equilateral triangle.

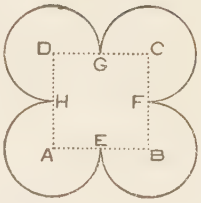
44. How far does the end of the minute hand of a clock move in 20 minutes, if the hand is $3\frac{1}{2}$ inches long?

45. The diameter of a circle is 16 inches. What is the area of that portion of the circle outside the inscribed regular hexagon?



46. Using the vertices of a square whose side is 12, as centers, and radii equal to 4, four quadrants are described within the square. Find the perimeter and the area of the figure thus formed.

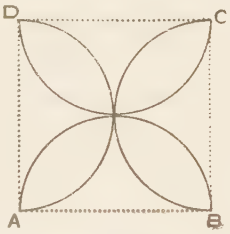
47. Using the four vertices of a square whose side is 12 as centers, and radii equal to 6, four arcs are described without the square (see figure). Find the perimeter and the area of the figure bounded by these four arcs.



48. Using the vertices of an equilateral triangle whose side is 16 as centers and radii equal to 8, three arcs are described within the triangle. Find the perimeter and the area of the figure bounded by these arcs. Do the same if the three arcs are described *without* the triangle.

49. Using the vertices of a regular hexagon, whose side is 20, as centers and radii equal to 10, six arcs are described within the hexagon. Find the perimeter and the area of the figure bounded by these arcs. Do the same if the six are described without the hexagon.

50. If semicircumferences are described within a square, with side 8 inches, upon the four sides as diameters, find the areas of the four lobes bounded by the eight quadrants. Find the area of any one.



In the following exercises let n = number of sides of the regular polygon; s = length of side; r = apothem; R = radius; K = area.

51. If $n = 3$, show that $s = R\sqrt{3}$; $r = \frac{1}{2}R$; $K = \frac{3R^2\sqrt{3}}{4} = 3r^2\sqrt{3}$.

52. If $n = 4$, show that $s = R\sqrt{2} = 2r$; $K = 2R^2 = 4r^2$.

53. If $n = 6$, show that $s = R = \frac{2r\sqrt{3}}{3}$; $K = \frac{3R^2\sqrt{3}}{2} = \frac{3s^2\sqrt{3}}{2} = 2r^2\sqrt{3}$.

54. If $n = 8$, show that $s = R\sqrt{2-\sqrt{2}} = 2r(\sqrt{2}-1)$; $r = \frac{R}{2}\sqrt{2+\sqrt{2}}$;

$R = r\sqrt{4-2\sqrt{2}}$; $K = 2R^2\sqrt{2} = 8r^2(\sqrt{2}-1)$.

55. If $n = 10$, show that $s = \frac{R}{2}(\sqrt{5}-1)$; $r = \frac{R}{4}\sqrt{10+2\sqrt{5}}$.

56. If $n = 5$, show that $s = \frac{R}{2}\sqrt{10-2\sqrt{5}}$; $r = \frac{R}{4}(\sqrt{5}+1)$.

57. If $n = 12$, show that $s = R\sqrt{2-\sqrt{3}} = 2r(2-\sqrt{3})$;

$R = 2r\sqrt{2-\sqrt{3}}$; $r = \frac{R}{2}\sqrt{2+\sqrt{3}}$; $K = 12r^2(2-\sqrt{3}) = 3R^2$

58. The apothem of a regular hexagon is $18\sqrt{3}$ inches. Find its side and area. Find the area of the circle circumscribed about it.

59. What is the radius of a circle whose area is doubled by increasing the radius 10 feet?

60. If an 8-inch pipe will fill a cistern in 3 hours 20 minutes, how long will it require a 2-inch pipe to fill it?

61. The radius of a circle is 12 meters. Find:

(*a*) The area of the inscribed square.
(*b*) The area of the inscribed equilateral triangle.
(*c*) The area of the inscribed regular hexagon.
(*d*) The area of the inscribed regular dodecagon.
(*e*) The area of the circumscribed square.
(*f*) The area of the circumscribed equilateral triangle.
(*g*) The area of the circumscribed regular hexagon.
(*h*) The area of the circumscribed regular dodecagon.

62. The radius of a circle is 18. Find:

(*a*) The side and the apothem of the inscribed square.
(*b*) The side and the apothem of the inscribed equilateral triangle.
(*c*) The side and the apothem of the inscribed regular hexagon.
(*d*) The area of the inscribed square.
(*e*) The area of the inscribed equilateral triangle.
(*f*) The area of the inscribed regular hexagon.
(*g*) The area of the inscribed regular octagon.
(*h*) The area of the circumscribed regular hexagon.

63. Prove that the area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and the circumscribed equilateral triangles. [Find the three areas in terms of R.]

64. AB is one side of an inscribed equilateral triangle, and C is the midpoint of AB. If AB is prolonged to O making BO equal to BC, and OT is drawn tangent to the circle at T, OT is $\frac{3}{2}$ the radius.

65. A square, an equilateral triangle, a regular hexagon, and a circle all have the same area, namely 5544 sq. ft. Which figure has the least perimeter? the greatest?

66. A square, an equilateral triangle, a regular hexagon, and a circle all have the same perimeter, namely 396 in. Find their areas and compare them.

67. The circumferences of two concentric circles are 330 in. and 440 in. respectively. Find the radius of another circle equivalent to the ring between these two circles.

ORIGINAL CONSTRUCTIONS

It is required:

1. To circumscribe a regular hexagon about a given circle.

2. To circumscribe an equilateral triangle about a given circle.

3. To circumscribe a regular decagon about a given circle; a regular 16-gon; a regular 24-gon; a square.

4. To construct an angle of 36°; of 18°; of 72°; of 24°; of 6°; of 48°; of 96°.

5. To construct a regular hexagon upon a given line as a side.

6. To construct a regular decagon upon a given line as a side.

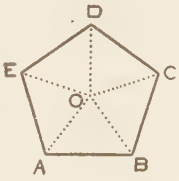
7. To construct a regular octagon upon a given line as a side.

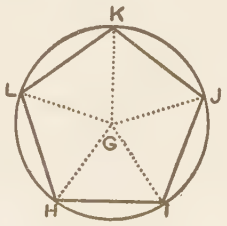
8. To construct a regular pentagon upon a given line as a side.

9. To construct a square with double the area of a given square.

10. To inscribe in a given circle a regular polygon similar to a given regular polygon.

Construction: From the center of the polygon draw radii. At the center of the circle construct $\measuredangle =$ to these central $\measuredangle$ of the polygon Draw chords. Etc.





11. To construct a regular pentagon which shall have double the area of a given regular pentagon.

12. To construct a circumference equal to the sum of two given circumferences.

13. To construct a circumference which shall be three times a given circumference.

14. To construct a circumference equal to the difference of two given circumferences.

15. To construct a circle whose area shall be five times a given circle.

16. To construct a circle equal to the sum of two given circles; another, equal to their difference.

17. To construct a circle whose area shall be half a given circle.

18. To bisect the area of a given circle by a concentric circle.

19. To divide a given circumference into two parts which shall be in the ratio of 3 : 7; into two other parts which shall be in the ratio of 5 : 7; into still two other parts, in the ratio of 8 : 7.

MAXIMA AND MINIMA

454. Of geometrical magnitudes that satisfy a given condition (or given conditions) the greatest is called the **maximum**, and the least, the **minimum.**

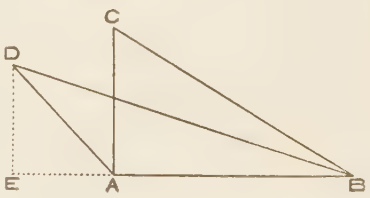
Thus, of all chords that can be drawn through a given point within a circle, the diameter is the maximum, and the chord perpendicular to the diameter at the point is the minimum.

Isoperimetric figures are figures having equal perimeters.

Proposition XVIII. Theorem

455. Of all triangles having two given sides, that in which these sides form a right angle is the maximum.

Given: $\triangle ABC$ and $\triangle ABD$ having AB common, and $AC =$ to AD; $\angle CAB$ a rt. $\angle$ and $\angle DAB$ not a right $\angle$.



To Prove: $\triangle ABC > \triangle ABD$.

Proof: Draw altitude DE.

Now	$AD > DE$	(87).
	$\therefore AC > DE$	(Ax. 6).

Multiply each member by $\frac{1}{2} AB$.

Then	$\frac{1}{2} AB \cdot AC > \frac{1}{2} AB \cdot DE$	(Ax. 10).
Now	$\frac{1}{2} AB \cdot AC =$ area $\triangle ABC$	(364),
And	$\frac{1}{2} AB \cdot DE =$ area ABD	(?).
Therefore	$\triangle ABC > ABD$	(Ax. 6).

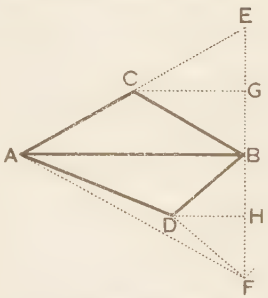
Q.E.D.

This theorem may be stated thus: *Of all triangles having two given sides, that triangle whose third side is the diameter of the circle which circumscribes it is the maximum.*

456. Corollary. Of all n-gons having $n - 1$ sides given, that polygon whose nth side is the diameter of a circle which circumscribes the polygon is the maximum.

Proposition XIX. Theorem

457. **Of all isoperimetric triangles having the same base, the isosceles triangle is the maximum.**



Given: $\triangle ABC$ and ABD isoperimetric, having the same base, AB, and $\triangle ABC$ isosceles.

To Prove: $\triangle ABC > \triangle ABD$.

Proof: Prolong AC to E, making $CE =$ to AC, and draw BE. Using D as a center and BD as a radius, describe an arc cutting EB prolonged, at F. Draw CG and $DH \parallel$ to AB, meeting EF at G and H respectively. Draw AF.

With C as a center and AC, BC or EC as a radius, the $\odot$ described will pass through A, B, and E (Hyp. and Const.).

$\therefore \angle ABE =$ rt. $\angle$ (240).

That is, AB is $\perp$ to EF.

Hence CG and DH are $\perp$ to EF (64).

$AC + CE = AC + CB = AD + DB = AD + DF$ (Hyp. and Const.).

That is, $AE = AD + DF$ (Ax. 1).

But $AD + DF > AF$ (Ax. 12).

$\therefore AE > AF$ (Ax. 6).

$\therefore BE > BF$ (88, IV).

And $\frac{1}{2} BE > \frac{1}{2} BF$ (Ax. 10).

Now $BG = \frac{1}{2} BE$, and $BH = \frac{1}{2} BF$ (85).

$\therefore BG > BH$ (Ax. 6).

Mult. by $\frac{1}{2} AB$, $\frac{1}{2} AB \cdot BG > \frac{1}{2} AB \cdot BH$ (Ax. 10).

But $\frac{1}{2} AB \cdot BG =$ area $\triangle ABC$ (364).

And $\frac{1}{2} AB \cdot BH =$ area $\triangle ABD$ (?).

Substituting, $\triangle ABC > \triangle ABD$ (Ax. 6).

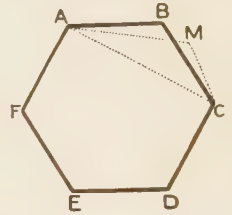
Q.E.D.

458. Corollary. Of isoperimetric triangles the equilateral triangle is the maximum.

[Any side may be considered the base.]

Proposition XX. Theorem

459. Of isoperimetric polygons having the same number of sides the maximum is equilateral.



Given: Polygon AD, the maximum of all polygons having the same perimeter and the same number of sides.

To Prove: $AB = BC = CD = DE =$ etc.

Proof: Draw AC and suppose AB not $=$ to BC.

On AC as base, construct $\triangle ACM$ isoperimetric with $\triangle ABC$ *and isosceles;* that is, make $AM = CM$.

Then $\triangle ACM > \triangle ABC$ (457).

Add to each member, the polygon $ACDEF$.

$\therefore$ polygon $AMCDEF >$ polygon AD (Ax. 7).

But the polygon AD is maximum (Hyp.).

$\therefore$ AB cannot be unequal to BC as we supposed (because that results in an impossible conclusion).

Hence $AB = BC$. Likewise it is proved that $BC = CD =$ etc.

Q.E.D.

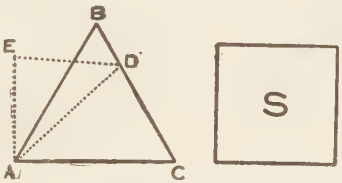
460. Corollary. Of isoperimetric polygons having the same number of sides the regular polygon is maximum.

Proof: Only one such polygon is maximum, and the maximum is equilateral. (459.)

It can also be inscribed in a circle and is therefore regular. (403.)

PROPOSITION XXI. THEOREM

461. **Of isoperimetric regular polygons, the polygon having the greatest number of sides is maximum.**



Given: Equilateral $\triangle ABC$ and square S, having the same perimeter.

To Prove: Square $S > \triangle ABC$

Proof: Take D, any point in BC, and draw AD. On AD as base, construct isosceles $\triangle ADE$, isoperimetric with $\triangle ABD$.

Now $\triangle AED > \triangle ABD$ (457).

Adding $\triangle ADC$ to each member, $AEDC > \triangle ABC$ (Ax. 7).

$AEDC$ is isoperimetric with $\triangle ABC$ and S (Hyp. and Const.).

Hence $S > AEDC$ (460).

$\therefore S > \triangle ABC$ (Ax. 11).

Similarly, we may prove that an isoperimetric regular pentagon is greater than S; and an isoperimetric regular hexagon is greater than this pentagon, etc.

Therefore the regular polygon having the greatest number of sides is maximum. Q.E.D.

462. COROLLARY. **Of all isoperimetric plane figures the circle is the maximum.**

Ex. 1. Of isoperimetric triangles, the maximum is equilateral.

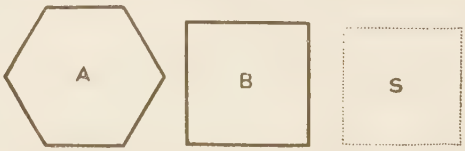
Ex. 2. Of all right triangles that can be constructed upon a given hypotenuse, which is maximum? Why?

Ex. 3. Of all triangles having a given base and a given vertex angle, the isosceles is the maximum.

Ex. 4. Of all mutually equilateral polygons, that which can be inscribed in a circle is the maximum.

Proposition XXII. Theorem

463. Of equal regular polygons the perimeter of the polygon having the greatest number of sides is the minimum.



Given: Any two equal regular polygons, A and B, A having the greater number of sides.

To Prove: the perimeter of $A <$ the perimeter of B.

Proof: Construct regular polygon S, similar to B and isoperimetric with A.

Then	$A > S$	(460).
But	$A = B$	(Hyp.).
	$\therefore B > S$	(Ax. 6).
Hence	the perimeter of $B >$ perimeter of S	(376).
But	the perimeter of $S =$ perimeter of A	(Const.).
	$\therefore$ perimeter of $B >$ perimeter of A	(Ax. 6).

That is, the perimeter of $A <$ the perimeter of B. Q.E.D.

464. Corollary. Of all equal plane figures the circle has the minimum perimeter.

Historical Note. René Descartes was born near Tours, France, in 1596. He was a man of wonderful intellect. He simplified and generalized the notation of algebra and introduced the use of exponents as now employed. The restriction of final letters of the alphabet to represent unknown quantities is also due to him.

Descartes

Descartes was the first to adapt algebra to geometry, showing that geometrical figures can be represented by algebraic equations. On this general truth he based the development of analytical geometry which is known by his own name, as *Cartesian* geometry. He gave a large part of his life to original and creative work in mathematics, philosophy, physics and astronomy.

ORIGINAL EXERCISES

1. Of all equal parallelograms having equal bases, the rectangle has the minimum perimeter.

2. Of all lines drawn between two given parallels (terminating both ways in the parallels), which is the minimum? Prove.

3. Of all straight lines that can be drawn on the ceiling of a room 12 feet long and 9 feet wide, what is the length of the maximum?

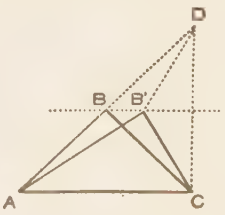
4. Find the areas of an equilateral triangle, a square, a regular hexagon, and a circle, the perimeter of each being 264 inches. Which is maximum? What theorem does this exercise illustrate?

5. Find the perimeters of an equilateral triangle, a square, a regular hexagon, and a circle, if the area of each is 1386 square feet. Which perimeter is the minimum? What theorem does this exercise illustrate?

6. Of isoperimetric rectangles which is maximum?

7. Divide a given line into two parts such that their product (rectangle) is maximum.

8. Of all equal triangles having the same base, the isosceles triangle has the minimum perimeter.



To Prove: The perimeter of $\triangle ABC <$ the perimeter of $\triangle AB'C$.

Proof: $AD < AB' + B'D$; etc.

9. Of all rectangles inscribed in a circle, which is maximum? Prove.

10. Of all rectangles inscribed in a semicircle, which is maximum? Prove.

11. Of all equal rectangles, the square has the minimum perimeter.

12. Of all triangles having a given base and a given vertex angle, the isosceles triangle has the maximum area.

13. Of all triangles having a given altitude and a given vertex angle, the isosceles triangle is the minimum.

14. Of all triangles that can be inscribed in a given circle, the equilateral triangle has the maximum area.

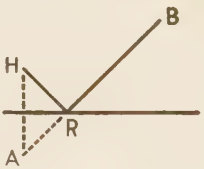
15. The cross section of a bee's cell is a regular hexagon. Would this be the most economical for the bee (that is, would he use the least wax) if one cell in a hive were all he were to fill? Considering also the adjoining cells, does the form of the regular hexagon require the least wax? Explain. Does it also permit the storing of the most honey? Why?

16. Prove, by the method employed in 461, that a regular hexagon is greater than an isoperimetric square.

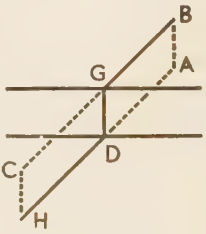
17. Answer the questions of exercise 65 on page 252, without any computation. Give reasons.

18. Compare the areas of the figures mentioned in Ex. 66, page 252, without performing any computation.

19. A farmer's house and barn are near a river. He wishes to lay from the house to the barn, the shortest possible path which shall reach to the water's edge. Draw a plan of the situation and the desired path and prove it the minimum.



20. A farmer's house and barn are on opposite sides of a straight stream. He wishes to lay a road from one to the other, and erect a bridge across the stream at right angles to the banks, by constructing the shortest possible track. Draw a plan of the situation and the desired road, and prove it the minimum.



21. A man has material to build 1000 yards of fence. With this he desires to inclose the largest possible yard for his poultry. What will be the shape and the area of his yard?

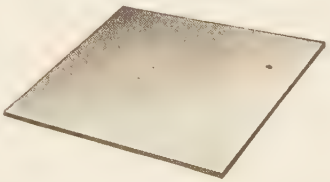
SOLID GEOMETRY

BOOK VI

LINES, PLANES, AND ANGLES IN SPACE

465. A **solid** is any limited portion of space. The boundaries of a solid are **surfaces.**

A **plane** is a surface in which, if any two points are taken, the straight line connecting them lies wholly in that surface.



Solid geometry is a science that treats of magnitudes, the parts of which are not all in the same plane.

466. The **intersection** of two surfaces is the line, or the lines, all of whose points lie in both surfaces. The intersection of a line and a surface is the point, or the points, common to both the line and the surface. The **foot of a line** intersecting a plane is their point of intersection.

467. A **straight line** is **perpendicular** to a plane if the line is perpendicular to every straight line in the plane drawn through its foot.

A **normal** is a straight line perpendicular to a plane.

468. A **straight line** is **parallel** to a plane if the line and the plane never meet, when indefinitely extended. A **straight line** is **oblique** to a plane if it is neither perpendicular nor parallel to the plane. Two **planes** are **parallel** if they never meet when indefinitely extended.

469. The **projection** of a **point** upon a plane is the foot of the perpendicular from the point to the plane.

The **projection** of a **line** upon a plane is the line formed by the projections of all the points of the given line.

470. A plane is **determined** if its position is fixed and if that position can be occupied by only one plane.

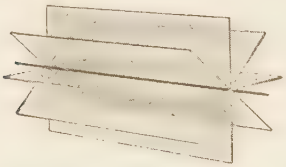
PRELIMINARY THEOREMS

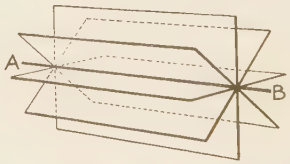
471. If two points of a straight line are in a plane, the whole line is in the plane. (Def. 465.)

472. A straight line can intersect a plane in only one point. (471.)

473. If a line is perpendicular to a plane, it is perpendicular to every line in the plane drawn through its foot. (467.)

474. Through one straight line any number of planes may be passed.





Because, if we consider a plane containing a line AB to revolve about AB, it may occupy an indefinitely great number of positions. Each of these will be a different plane containing AB.

475. Through a fixed straight line and an external point a plane can be passed.

Because, if we pass a plane containing this line AB, it may be revolved about AB until it contains the given point.

476. A straight line and an external point determine a plane. (See 475, 470.)

477. Three points not in a straight line determine a plane.

Because two of the points may be joined by a line; then this line and the third point determine a plane. (See 476.)

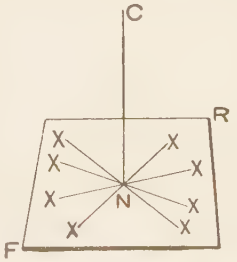
478. Two parallel lines determine a plane. (21, 476.)

479. Two intersecting straight lines determine a plane.

Because one of these lines and a point in the second line determine a plane; and this plane contains the second line. (476, 471.)

480. If two planes are parallel, no line in the one can meet any line in the other. (Def. 468.)

NOTE. A plane is represented to the eye by a quadrilateral. In some positions it appears to be a parallelogram, and in others, a trapezoid. The eye, however, must be aided by the imagination in really understanding the diagrams of solid geometry. Thus, in the adjoining figure, the line CN is perpendicular to the plane FR, and to every line in FR drawn through N. Consider several lines drawn through a point on the floor, and a cane, CN, occupying a vertical position, so that it is perpendicular to all these lines. Then every angle CNX is a right angle, though to the unskilled eye they do not all appear to be right angles in the diagram. The object of all geometrical diagrams is that the eye may assist the mind in grasping truths or in developing logical demonstrations, and the student should thoroughly examine every figure until he completely understands the relative positions of its parts. A photograph, like a geometrical diagram, represents three dimensions in a plane, and we should be as familiar with the significance of one as with the other.



When, during the process of a demonstration or elsewhere, it becomes necessary to employ a plane not already indicated, it is customary to pass such a plane, or to conceive it constructed.

Ex. 1. How many planes can be passed through two points? through three points in the same straight line?

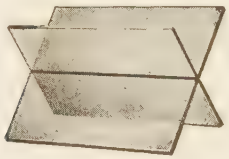
Ex. 2. Hold two pencils (representing lines) so that a plane can be passed containing both; so that no plane can be passed containing both.

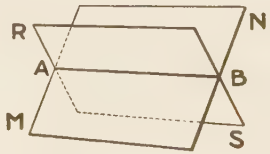
THEOREMS AND DEMONSTRATIONS

POINTS, LINES, AND PLANES

PROPOSITION I. THEOREM

481. If two planes intersect, their intersection is a straight line.





Given: Intersecting planes *MN* and *RS*.

To Prove: Their intersection is a straight line.

Proof: Suppose *A* and *B* are two points common to both planes.

Draw straight line *AB*.

Now *AB* is in plane *RS* (471).

And *AB* is in plane *MN* (?).

That is, *AB* is common to both planes.

Again, if there were a point outside of *AB* in both planes, these planes would coincide (476).

Hence *AB* contains *all points* common to the two given planes. ∴ *AB* is the intersection (466).

That is, the intersection of the two planes is a straight line. Q.E.D.

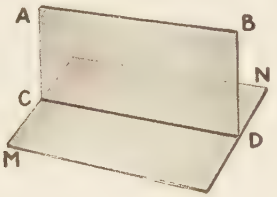
Ex. 1. Do any two planes intersect? Explain.

Ex. 2. What is meant by the statement "Two planes determine a line"? Is this universally true?

Ex. 3. What kind of lines are the folds in your letter paper? in a pamphlet? in the edges of a box or a brick? Explain.

Proposition II. Theorem

482. If two straight lines are parallel, a plane containing one, and only one, is parallel to the other line.



Given: ∥ lines AB and CD; plane MN containing CD.

To Prove: plane MN ∥ to line AB.

Proof: AB and CD are in the same plane AD (21).

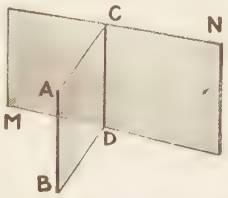
Plane AD intersects plane MN in CD (481).

Now as AB cannot meet MN in CD (hyp.) it can never meet MN.

$\therefore$ AB is ∥ to MN (468). Q.E.D.

Proposition III. Theorem

483. If a straight line is parallel to a plane, and another plane containing this line intersects the given plane, the intersection is parallel to the given line.



Given: AB ∥ to MN; plane AD containing AB and intersecting plane MN in CD.

To Prove: AB ∥ to CD.

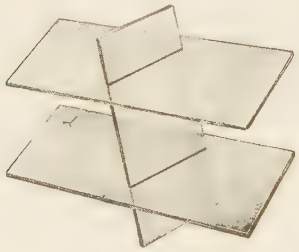
Proof: AB and CD are in the same plane AD (Hyp.).

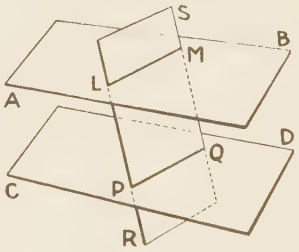
Now as AB cannot meet MN (hyp.) it can never meet CD in MN.

$\therefore$ AB is ∥ to CD (21). Q.E.D.

Proposition IV. Theorem

484. The intersections of two parallel planes by a third plane are parallel lines.





Given: ∥ planes AB and CD cut by plane RS in lines LM and PQ.

To Prove: LM ∥ to PQ.

Proof: LM and PQ are in the same plane RS (Hyp.).

Also LM and PQ can never meet (480).

$\therefore$ LM is ∥ to PQ (21).

Q.E.D.

Ex. 1. Hold a pencil parallel to the blackboard, so that its shadow falls on the blackboard. Is this shadow parallel to the pencil? Why?

Ex. 2. Can a plane intersect two planes that are not parallel so that the intersections are parallel? Illustrate your answer by passing a plane across the room so that it cuts the end and a side of the room. How must this plane be passed so that the intersections are parallel lines?

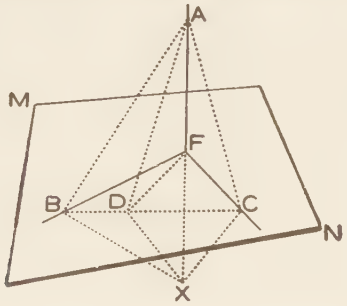
Ex. 3. Draw a line on the blackboard that will never meet the plane of the ceiling. How was it drawn?

Ex. 4. Draw a line on the blackboard and one on the floor that will meet if extended. How must these lines be drawn?

Ex. 5. When will a line on the ceiling be parallel to a line on the floor?

Proposition V. Theorem

485. A straight line perpendicular to each of two straight lines at their intersection is perpendicular to the plane of the lines.



Given: $AF \perp$ to BF and CF at F; plane MN containing BF and CF.

To Prove: $AF \perp$ to plane MN.

Proof: In plane MN draw BC; draw also DF from F to any point, D, in BC.

Prolong AF to X, making $FX =$ to AF, and draw AB, AD, AC, BX, DX, CX.

BF and CF are $\perp$ bisectors of AX (Hyp. and Const.).

In $\triangle ABC$ and BXC, $AB = BX$ and $AC = CX$ (80).

$BC = BC$ (?).

$\therefore \triangle ABC \cong \triangle BXC$ (78).

Also in $\triangle ABD$ and BXD, $\angle ABC = \angle CBX$ (27).

$BD = BD$ (?).

And $AB = BX$ (?).

$\therefore \triangle ABD \cong \triangle BXD$ (52).

$\therefore AD = DX$ (?).

Hence DF is $\perp$ to AX (83).

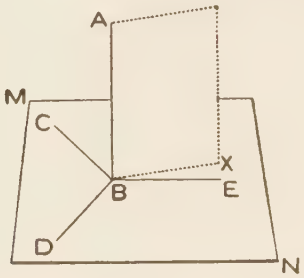
That is, AF is $\perp$ to all lines in MN through F.

$\therefore AF$ is $\perp$ to plane MN (467).

Q.E.D.

Proposition VI. Theorem

486. All straight lines perpendicular to a line at one point are in one plane, which is perpendicular to this line at this point.



Given: $AB \perp$ to BC, BD, BE, etc.; plane MN containing BC and BD.

To Prove: BE is in the plane MN and MN is $\perp$ to AB at B.

Proof: Pass plane AE containing AB and BE, and intersecting plane MN in line BX.

Now AB is $\perp$ to plane MN (485).

That is, plane MN is $\perp$ to AB

$\therefore$ AB is $\perp$ to BX (473).

But AB is $\perp$ to BE (Hyp.).

That is, BX and BE are both in plane AX and $\perp$ to AB at B.

$\therefore$ BX and BE coincide (43).

That is, BE is in plane MN. Q.E.D.

Ex. 1. Can a line be perpendicular to two other lines if these two do not intersect? Illustrate.

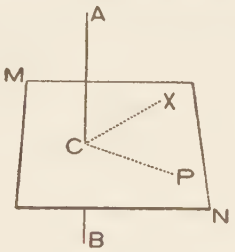
Ex. 2. If two lines are perpendicular to a third line are they necessarily in the same plane? Are the two lines necessarily parallel? Illustrate.

Ex. 3. Give a reason why the "corners" of a building are perpendicular to the horizontal plane of a level street.

487. COROLLARY. **Through a point in a straight line one plane can be passed perpendicular to the line, and only one.** (486.)

PROPOSITION VII. THEOREM

488. Through an external point one plane can be passed perpendicular to a given straight line, and only one.



Given: The line AB and point P outside of AB.

To Prove: Through P, one plane can be passed $\perp$ to AB, and only one.

Proof: I. Draw from P, $PC \perp$ to AB, and at C draw CX, another line $\perp$ to AB.

PC and CX determine a plane MN (479).

Plane MN contains P and is $\perp$ to AB (485).

II. Only one line $\perp$ to AB can be drawn from P (54).

And only one plane $\perp$ to AB can be passed at C (487).

That is, MN is the only plane $\perp$ to AB that can be passed through P. Q.E.D.

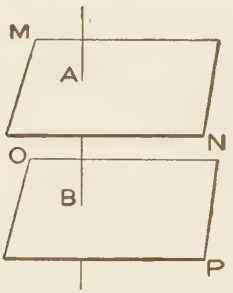
Ex. 1. Considered as lines, why are the spokes of a cart wheel perpendicular to the axle?

Ex. 2. As the hand of a clock revolves, what may it be said to describe, if it is considered of indefinite length? Why?

Ex. 3. Illustrate Proposition VI by revolving a carpenter's square, holding one straight edge against the wall or the floor.

Proposition VIII. Theorem

489. Two planes perpendicular to the same straight line are parallel.



Given: Planes *MN* and *OP* ⊥ to *AB*.

To Prove: Plane *MN* ∥ to plane *OP*.

Proof: If the planes *MN* and *OP* are not ∥, they will meet when sufficiently extended (Def. 468).

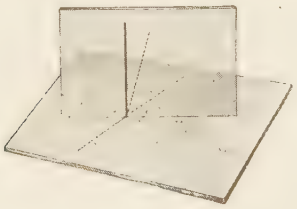
Then there would be two planes from the same point ⊥ to *AB* (⊥ by hyp.).

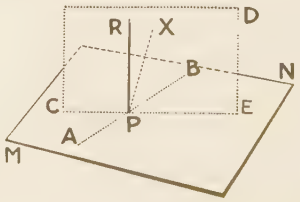
But this is impossible (488).

∴ the planes never meet and are parallel (468). Q.E.D.

Proposition IX. Theorem

490. At a given point in a plane one line can be drawn perpendicular to the given plane, and only one.





Given: Plane *MN* and point *P* within it.

To Prove: One line can be drawn $\perp$ to plane MN at P, and only one.

Proof: I. In plane MN draw any line AB, through P

Suppose plane CD is passed $\perp$ to AB at P, meeting the plane MN in CE.

In plane CD draw $PR \perp$ to CE, from P.

Now	AB is $\perp$ to plane CD	(Const.).
	$\therefore AB$ is $\perp$ to PR	(473).
	PR is $\perp$ to CE	(Const.).
	$\therefore PR$ is $\perp$ to plane MN	(485). Q.E.D.

II. Suppose another line PX to be $\perp$ to plane MN at P.

Then	PX and PR determine a plane CD	(479).
And	plane CD intersects plane MN in line CE	(481).
Then	PX and PR would both be $\perp$ to CE at P	(473).
But	this is impossible	(43).

That is, PX and PR coincide and PR is the only $\perp$ to plane MN at P. Q.E.D.

Ex. 1. How many positions can a flagpole occupy without being perfectly erect? How many positions may it assume and be perfectly erect?

Ex. 2. Name all the right angles at P, in figure of 490, and tell why each is a right angle.

Ex. 3. What information can the mason or the surveyor obtain from a plumb bob? Does he obtain this information when the bob is swinging or when it is stationary?

Ex. 4. In transplanting a tree to a horizontal lawn a gardener may use a carpenter's square to make certain that the tree is perpendicular to the lawn. In how many different positions must he place the square against the tree to ascertain its erectness? Why?

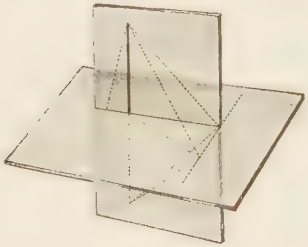
Ex. 5. In the diagram of 490, if PX is in plane CD, is it perpendicular to CE? Why?

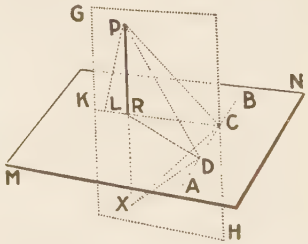
Ex. 6. In the same diagram, if PX is perpendicular to CE, is it in plane CD? Why?

Ex. 7. How many planes are determined by four random fixed points (that is, not all in one plane)?

Proposition X. Theorem

491. Through a given external point one line can be drawn perpendicular to a given plane, and only one.





Given: Plane MN and point P outside of it.

To Prove: One line can be drawn through $P \perp$ to plane MN, and only one.

Proof: I. In plane MN draw any line AB.

Suppose a plane GH is passed through $P \perp$ to AB, meeting plane MN in KC, and AB at C.

In plane GH draw $PR \perp$ to KC and prolong PR to X, making $RX = PR$.

Draw RD to any point in AB, except C.

Draw PC, PD, CX, DX.

Now	RC is $\perp$ to PX at its midpoint	(Const.).
Also	AB is $\perp$ to plane GH	(Const.).
Hence	$\measuredangle DCP$ and DCX are rt. $\measuredangle$	(473).
In rt. $\triangle DCP$ and DCX,	$DC = DC$	(?).
	$PC = CX$	(80).
	$\therefore \triangle DCP \cong \triangle DCX$	(53).
	$\therefore PD = XD$	(?).
	$\therefore RD$ is $\perp$ to PX	(83).

That is, PR is $\perp$ to RC and RD, in plane MN.

$\therefore PR$ is $\perp$ to plane MN from P (485). Q.E.D.

II. Suppose there is another line PL, $\perp$ to plane MN from P.

Then PR and PL determine a plane GH (479).

This plane intersects plane MN in KC (481).
PR and PL would then both be $\perp$ to KC (473).
But this is impossible (54).

That is, PR and PL coincide, and therefore PR is the only line $\perp$ to plane MN from P. Q.E.D.

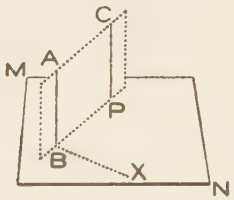
Ex. Name all the right angles at C in the figure of 491.

492. Corollary. **If a plane is perpendicular to a line in another plane, any line in the first plane perpendicular to the intersection of the planes is perpendicular to the second plane.**

Proof: Identical with the proof of 491, I.

Proposition XI. Theorem

493. If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other also.



Given: Plane $MN \perp$ to line AB, and $AB \parallel$ to CP.

To Prove: $CP \perp$ to plane MN.

Proof: AB and CP determine a plane. (478.)
Pass this plane BC, intersecting plane MN in line BP.
Draw $BX \perp$ to BP, in plane MN.

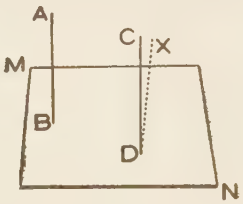
AB is $\perp$ to BX (473).
$\therefore$ BX is $\perp$ to plane BC (485).
But AB is $\perp$ to BP (473).
$\therefore$ BP is $\perp$ to CP (64).

That is, plane BC is $\perp$ to BX, and CP, in plane BC, is $\perp$ to the intersection BP.

$\therefore$ CP is $\perp$ to plane MN (492). Q.E.D.

Proposition XII. Theorem

494. Two lines perpendicular to the same plane are parallel.



Given: Lines AB and $CD \perp$ to plane MN.

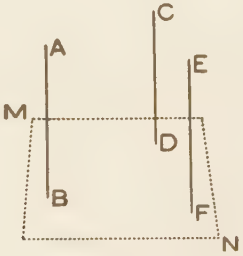
To Prove: $AB \parallel$ to CD.

Proof: Through D, the foot of CD, draw $DX \parallel$ to AB.

Then	DX is $\perp$ to plane MN	(493).
But	CD is $\perp$ to plane MN at D.	(Hyp.)
	$\therefore$ DX and DC coincide	(490).
That is,	AB is $\parallel$ to CD.	Q.E.D.

Proposition XIII. Theorem

495. Two straight lines that are parallel to a third straight line are parallel to each other.



Given: Lines CD and EF each $\parallel$ to AB.

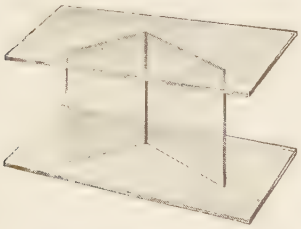
To Prove: $CD \parallel$ to EF.

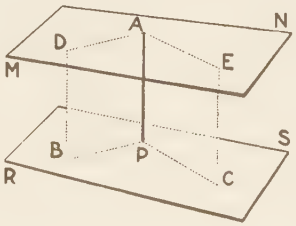
Proof: Suppose plane MN is passed $\perp$ to AB.

	$\therefore$ MN is $\perp$ to CD and to EF	(493).
	$\therefore$ CD is $\parallel$ to EF	(494). Q.E.D.

Proposition XIV. Theorem

496. A line perpendicular to one of two parallel planes is perpendicular to the other also.





Given: Plane $MN \parallel$ to plane RS; $AP \perp$ to plane RS.

To Prove: $AP \perp$ to plane MN.

Proof: Through AP pass any two planes, AB and AC, intersecting MN in AD and AE, and intersecting RS in PB and PC, respectively.

AD is $\parallel$ to PB, and AE is $\parallel$ to PC (484).

AP is $\perp$ to PB and PC (473).

$\therefore$ AP is $\perp$ to AD and AE (64).

$\therefore$ AP is $\perp$ to plane MN (485). Q.E.D.

497. Corollary. If two planes are each parallel to a third plane, they are parallel to each other.

Proof: Draw a line $\perp$ to the third plane.

This line is $\perp$ to each of the other planes (496).

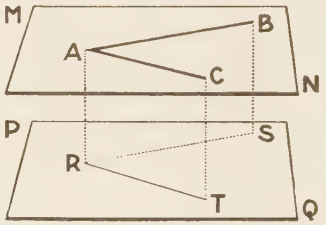
$\therefore$ The two planes are $\parallel$ (489).

Ex. 1. Can a line be perpendicular to both of two planes if they are not parallel? Prove.

Ex. 2. Are three lines that are perpendicular to the same plane necessarily parallel?

Proposition XV. Theorem

498. If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to the given plane.



Given: Intersecting lines AB and AC in plane MN; each line $\parallel$ to plane PQ.

To Prove: Plane $MN \parallel$ to plane PQ.

Proof: Draw $AR \perp$ to MN at A, meeting PQ at R.

Through AR and AB pass plane AS, and through AR and AC pass plane AT, intersecting plane PQ in RS and RT, respectively.

Now	AB is $\parallel$ to RS, and AC is $\parallel$ to RT	(483).
	AR is $\perp$ to AB and AC	(473).
	$\therefore$ AR is $\perp$ to RS and RT	(64).
Hence	AR is $\perp$ to plane PQ	(485).
	$\therefore$ plane MN is $\parallel$ to plane PQ	(489).

Q.E.D.

Ex. 1. Can one line be perpendicular to two other lines that do not intersect? How?

Ex. 2. Can one line be perpendicular to two other lines that do intersect? How?

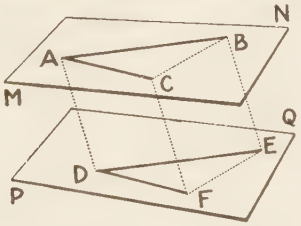
Ex. 3. Could 484 be quoted correctly as the reason that AB is parallel to RS in Proposition XV?

Ex. 4. Could 483 be quoted correctly as the reason that AD is parallel to BP in Proposition XIV?

Ex. 5. Prove Proposition XV by drawing AR perpendicular to PQ from A.

Proposition XVI. Theorem

499. **If two angles, not in the same plane, have their sides parallel each to each, and extending in the same directions from their vertices, the angles are equal and the planes are parallel.**



Given: $\angle BAC$ in plane MN and $\angle EDF$ in plane PQ; $AB \parallel$ to DE; $AC \parallel$ to DF, and extending in the same directions.

To Prove: I. $\angle BAC = \angle EDF$.

II. Plane $MN \parallel$ to plane PQ.

Proof: I. Take DE and AB equal, and DF and AC equal.

Draw AD, BE, CF, BC, EF.

	The figure $ABED$ is a $\square$	(129).
	$\therefore AD = BE$	(124).
Also	$ACFD$ is a $\square$	(?).
	$AD = CF$	(?).
	$\therefore BE = CF$	(?).
Again,	AD is $\parallel$ to BE and AD is $\parallel$ to CF	(120).
	$\therefore BE$ is $\parallel$ to CF	(495).
	$\therefore BCFE$ is a $\square$	(129).
Now in $\triangle ABC$ and DEF,	$AB = DE$; $AC = DF$	(Const.).
Also	$BC = EF$	(124).
	$\therefore \triangle ABC \cong \triangle DEF$	(78).
	$\therefore \angle BAC = \angle EDF$	(27).

Q.E.D.

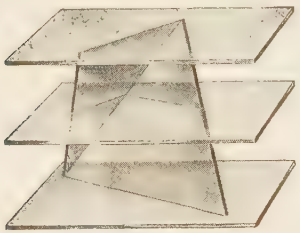
II. AB is $\parallel$ to plane PQ and AC is $\parallel$ to plane PQ (482).

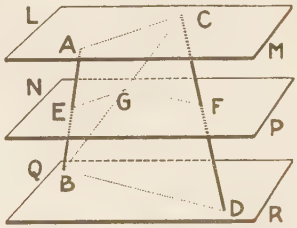
$\therefore$ plane MN is $\parallel$ to plane PQ (498).

Q.E.D.

Proposition XVII. Theorem

500. If three parallel planes intersect two straight lines, the corresponding intercepts are proportional.





Given: Parallel planes, LM, NP, QR, intersecting line AB at A, E, B, and CD at C, F, D, respectively.

To Prove: $AE : EB = CF : FD$.

Proof: Draw BC, meeting plane NP at G.

Through AB and BC pass a plane cutting LM in AC and NP in EG.

Through BC and CD pass a plane cutting NP in GF and QR in BD.

Now EG is $\parallel$ to AC and GF is $\parallel$ to BD (484).

$\therefore \dfrac{AE}{EB} = \dfrac{CG}{GB}$ and $\dfrac{CG}{GB} = \dfrac{CF}{FD}$ (293).

Consequently $AE : EB = CF : FD$ (Ax. 1).

Q.E.D.

Also $AE + EB : AE = CF + FD : CF$

Or $AE + EB : EB = CF + FD : FD$ (284).

$\therefore AB : AE = CD : CF$

Or $AB : EB = CD : FD$ (Ax. 6).

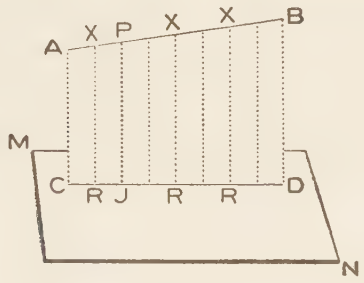
$\therefore \dfrac{AB}{CD} = \dfrac{AE}{CF} = \dfrac{EB}{FD}$ (282).

Ex. 1. If any number of lines which meet at a point are cut by two parallel planes, the corresponding intercepts are proportional.

Ex. 2. In the above diagram, why are not AC and BD parallel? Under what condition would EGF be a straight line?

Proposition XVIII. Theorem

501. **The projection of a straight line upon a plane is a straight line.***



Given: Line AB and plane MN.

To Prove: The projection of AB on MN is a straight line.

Proof: Draw $PJ \perp$ to plane MN from any point P, in AB.

AB and PJ determine a plane. (479).

Plane AD cuts plane MN in a straight line CD. (481).

Now in plane AD, draw $XR \parallel$ to PJ from X, any other point in AB.

XR is $\perp$ to plane MN. (493).

Now R is the projection of X. (469).

$\therefore$ CD is the projection of AB. (469).

That is, the projection of AB upon the plane MN is a straight line. Q.E.D.

Ex. 1. What is the length of the projection of a 5 ft. rod, inclined at an angle of 45°?

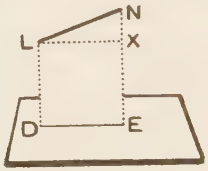
Ex. 2. A ladder 26 ft. long leans against the wall of a house, at a point 10 ft. from the ground. What is the length of the projection of the ladder on the ground?

502. Corollary. **A straight line and its projection upon a plane are in the same plane.**

*Except only if the given line is a normal to the given plane.

Proposition XIX. Theorem

503. A line not parallel to a plane is longer than its projection upon the plane.



Given: A plane and line LN not $\parallel$ to the plane, and DE the projection of LN upon the plane.

To Prove: $LN > DE$.

Proof: Draw LD and NE.

Draw $LX \perp$ to NE from L, in the plane LE.

LD and NE are $\perp$ to the plane. (Def. of projection, 469).

$LXED$ is a rectangle. (157).

Now $LN > LX$ (87).

But $LX = DE$ (124).

$\therefore LN > DE$ (Ax. 6). Q.E.D.

Ex. If a line is parallel to a plane, all points of the line are equally distant from the plane.

Proposition XX. Theorem

504. Of all lines that can be drawn to a plane from a point:

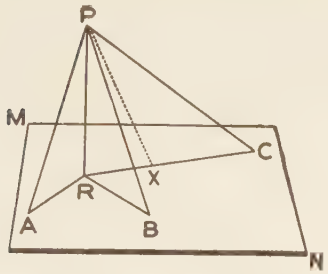
I. The perpendicular is the shortest.

II. Oblique lines having equal projections are equal.

III. Equal oblique lines have equal projections.

IV. Oblique lines having unequal projections are unequal, and the line having the greater projection is the longer.

V. Unequal oblique lines have unequal projections, and the longer line has the greater projection.



I. **Given:** Plane MN; point P; $PR \perp$ to MN; any other line from P to plane MN, as PA.

To Prove: $PR < PA$.

Proof: Draw AR.

Now	PR is $\perp$ to AR	(473).
And	PA is not $\perp$ to AR	(54).
	$\therefore PR < PA$	(87). Q.E.D.

II. **Given:** Oblique lines PA and PB whose projections, AR and BR, are equal.

To Prove: $PA = PB$.

Proof: The right $\triangle$ PAR and PRB are $\cong$ (?).

III. **Given:** Equal oblique lines PA and PB.

To Prove: Their projections, AR and BR, are equal.

Proof: The right $\triangle$ PAR and PRB are $\cong$ (?)

IV. **Given:** Oblique lines PC and PA; proj. $RC >$ proj. RA.

To Prove: $PC > PA$.

Proof: In $\triangle$ PRC, take on RC, $RX = RA$, and draw PX.

Now	$PC > PX$	(88, III).
But	$PA = PX$	(504, II).
	$\therefore PC > PA$	(Ax. 6). Q.E.D.

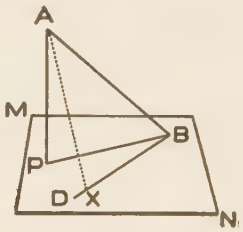
V. **Given:** Unequal oblique lines, $PC > PA$.

To Prove: Projection $RC >$ projection RA.

Proof: By method of exclusion (See 90).

Proposition XXI. Theorem

505. The acute angle that a line makes with its own projection upon a plane is the least angle that the line makes with any line of the plane.



Given: AB, any line meeting plane MN at B; BP, its projection upon MN; BD, any other line in MN, through B.

To Prove: $\angle ABP < \angle ABD$.

Proof: On BD take $BX = BP$ and draw AX.

In $\triangle APB$ and ABX,

	$AB = AB$	(?).
	$BP = BX$	(Const.).
But	$AP < AX$	(504, I).
	$\therefore \angle ABP < \angle ABD$	(92).

Q.E.D.

Ex. 1. If PR is 12 in., and AP 13 in., find the length of AR, in 504.

Ex. 2. Does the longer of two lines always have the longer projection on the same plane? Could they have equal projections? Illustrate.

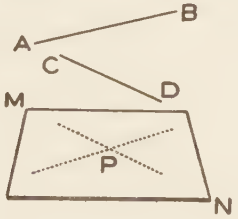
Ex. 3. With what line in a plane does a line oblique to that plane make the greatest angle?

Ex. 4. With what line in a plane does a line oblique to that plane make right angles?

Ex. 5. How do you construct the projection of a curved line upon a plane? When is this a straight line?

Proposition XXII. Theorem

506. Through a given point one plane can be passed parallel to any two given non-parallel lines in space, and only one.



Given: Point P; two lines, AB and CD.

To Prove: Through P one plane can be passed $\parallel$ to AB and CD, and only one.

Proof: I. Through P draw a line $\parallel$ to AB and another $\parallel$ to CD.

Pass a plane MN, containing these lines.

MN is $\parallel$ to both AB and CD (482).

II. Only one line can be drawn through P $\parallel$ to AB, and only one $\parallel$ to CD (Ax. 13).

$\therefore$ there is only one plane (479).

Q.E.D.

507. Corollary. If two lines are not in the same plane, one plane and only one can be passed through one of these lines parallel to the other.

[Through a point in one line draw a line $\parallel$ to the other line, etc.]

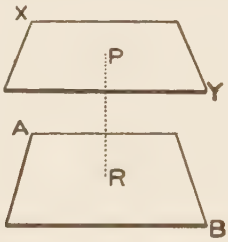
Ex. 1. Is Proposition XXII true if the given lines intersect? Is it true if they are parallel?

Ex. 2. Explain, so that a blind boy could understand, how to pass a plane through a given point and parallel to two pencils he may hold in his outstretched hands.

Ex. 3. Can two lines be parallel to a plane and not be parallel to each other? Illustrate, by means of pencils and the ceiling.

Proposition XXIII. Theorem

508. Through a given point one plane can be passed parallel to a given plane, and only one.



Given: (?). **To Prove:** (?).

Proof: I. Suppose PR is drawn $\perp$ to plane AB; and plane XY is passed $\perp$ to PR at P.

Then XY is $\parallel$ to AB (489).

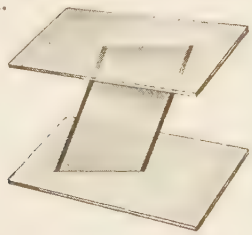
II. Only one line $\perp$ to AB can be drawn from P (491).

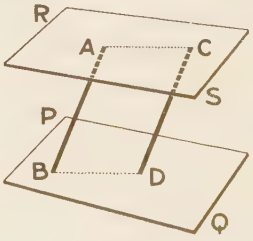
Only one plane $\perp$ to PR can be passed at P (487).

$\therefore$ only one plane can contain P and be $\parallel$ to AB. Q.E.D.

Proposition XXIV. Theorem

509. Parallel lines included between parallel planes are equal.





Given: (?). **To Prove:** (?).

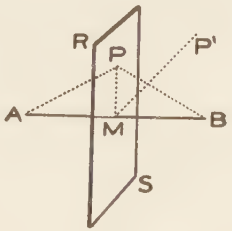
Proof: The plane determined by AB and CD intersects RS and PQ in lines AC and BD, which are $\parallel$ (484).

$\therefore$ $ABDC$ is a $\square$ (Def.).

Hence $AB = CD$ (124). Q.E.D.

Proposition XXV. Theorem

510. The plane perpendicular to a line at its midpoint is the locus of points in space equally distant from the extremities of the line.



Given: Plane $RS \perp$ to AB at its midpoint, M.

To Prove: Plane RS is the locus of points in space equally distant from A and B.

Proof: (1) Take P, any point in RS.

Draw PM, PA, PB.

Now PM is $\perp$ to AB (473).

$\therefore PA = PB$ (80).

That is, any point in RS is equally distant from A and B.

(2) Take P', any point outside of RS. Draw $P'M$.

Now $P'M$ is *not* $\perp$ to AB (486).

$\therefore$ P', any point outside of plane RS, is not equally distant from A and B (81).

$\therefore$ Plane RS is the locus of points in space, equally distant from A and B (246). Q.E.D.

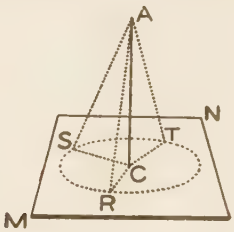
Ex. 1. How can you find a line in a plane such that each of its points is equally distant from two given points?

Ex. 2. How can you find that point in a given line which is equally distant from two given points?

Ex. 3. There are two definite lines in space. It is desired to find all points that are equally distant from the ends of one line and at the same time equally distant from the ends of the other. How can this be done?

Proposition XXVI. Theorem

511. The locus of points in space equally distant from all the points in the circumference of a circle is the line perpendicular to the plane of the circle at its center.



Given: (?). To Prove: (?).

Proof: I. Any point in AC is equally distant from all the points in the circumference of the circle (504, II).

II. Any point equally distant from all points of the circumference of the circle is in AC (504, III).

$\therefore$ AC is the required locus (246). Q.E.D.

Ex. 1. What is the locus of points equally distant from two given points?

Ex. 2. What is the locus of points equally distant from three given points?

Ex. 3. Draw a triangle on the blackboard and a definite straight line on the floor. Tell how to find the one point which is both equally distant from the vertices of the triangle and from the ends of the line. Is there *always* one point?

---

512. The **distance** from a point to a plane is the length of the perpendicular from the point to the plane.

Thus, the word "distance," referring to the shortest line from a point to a plane, implies the perpendicular.

The **inclination** of a line to a plane is the angle between the line and its projection upon the plane.

ORIGINAL EXERCISES

1. Through one straight line a plane can be passed parallel to any other straight line in space, and only one.

Through a point of the first line draw a line ∥ to the second.

2. Two parallel planes are everywhere equally distant.

3. If a line and a plane are parallel, another line parallel to the given line and through any point in the given plane lies wholly in the given plane.

Through the given line and the point P pass a plane cutting the given plane in PX. [Use 483.]

4. A straight line parallel to the intersection of two planes, but in neither, is parallel to both planes.

5. If two straight lines are parallel and two intersecting planes are passed, each containing one of the lines, the intersection of these planes is parallel to each of the given lines.

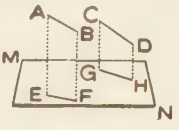
6. If three straight lines through a point meet the same straight line, these four lines all lie in the same plane.

7. If a straight line meets two parallel planes, its inclinations to the planes are equal.

8. Two parallel planes can be passed, each containing one of two given lines in space. Is this ever impossible?

9. If each of three straight lines intersects the other two, not at the same point, the three lines all lie in a plane.

10. The projections of two parallel lines on a plane are parallel.



Proof: AB is ∥ to CD (?). AE is ∥ to CG (?).
$\therefore$ planes AF and CH are ∥ (?); etc.

11. If two lines in space are equal and parallel, their projections on a plane are equal and parallel.

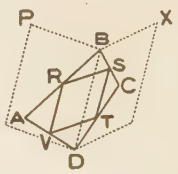
12. If a plane is parallel to one of two parallel lines, it is parallel to the other.

13. If a straight line and a plane are perpendicular to the same straight line, they are parallel.

14. Equal oblique lines drawn to a plane from one point have equal inclinations with the plane.

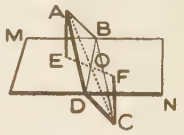
15. If a line and a plane are both parallel to the same line, they are parallel to each other.

16. Four points in space, A, B, C, D, are joined, and these four lines are bisected. Prove that the four lines joining (in order) the four midpoints of the first lines form a parallelogram.



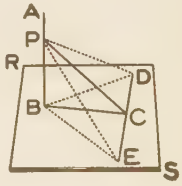
Proof: Pass plane DP through points A, D, B, and plane DX through points B, C, D, — these planes intersecting in BD. ST is $\parallel$ to BD and $= \frac{1}{2} BD$ (?); etc.

17. If a plane is passed containing a diagonal of a parallelogram and perpendiculars are drawn to the plane from the other vertices of the parallelogram, they are equal.



To Prove: $AE = CF$. **Proof**: Draw diagonal AC. Draw EO and OF in plane MN. EO, OF, and EOF are projections; etc.

18. If from the foot of a perpendicular to a plane, a line is drawn at right angles to any line in the plane, the line connecting this point of intersection with any point in the perpendicular is perpendicular to the line in the plane.



Given: $AB \perp$ to plane RS; $BC \perp$ to DE in the plane; PC drawn from C to P, in AB.

To Prove: PC is $\perp$ to DE.

Proof: Take $CD = CE$, draw PD, PE, BD, BE. BC is $\perp$ to DE at its midpoint (?). $\therefore BD = BE$ (?). $PD = PE$ (?) (504, II).

$\therefore PC$ is $\perp$ to DE (?) (83).

19. A line PB is perpendicular to a plane at B, and a line is drawn from B meeting any line DE, of the plane, at C. If PC is perpendicular to DE, BC is perpendicular to DE.

20. Are two planes that are parallel to the same straight line necessarily parallel?

21. If each of two parallel lines is parallel to a plane, is the plane of these lines also parallel to the given plane?

22. Is a three-legged stool always stable on the floor? Why? Is a four-legged chair always stable? Why?

23. What is the locus in space of points equally distant from two parallel planes? from two parallel lines?

24. What is the locus of points in space at a given distance from a given plane?

25. What is the locus of points in a plane at a given distance from an external point?

26. What is the locus of points in space equally distant from two points and equally distant from two parallel planes?

27. What is the locus of points in space equally distant from the vertices of a triangle?

28. What is the locus of all straight lines perpendicular to a given straight line at a given point?

29. What is the locus of all lines parallel to a given plane and drawn through a given point?

30. If the points in a line satisfy one condition and the points in a plane satisfy another condition, what will be true of their intersection? What will be true if they do not intersect?

31. If the points in one plane satisfy one condition and the points in another plane satisfy another condition, what is true of their intersection? What is true if the planes are parallel?

32. Construct a plane perpendicular to a given line at a given point in the line.

33. Construct a plane perpendicular to a given line through a given external point.

34. Construct a line perpendicular to a given plane through a given point in the plane; through a given external point.

35. Construct a plane parallel to a given plane through a given point.

36. Construct a number of equal oblique lines to a plane from a given external point.

37. Construct a line through a given point parallel to a given plane.

38. Construct through a given point a line parallel to each of two given intersecting planes.

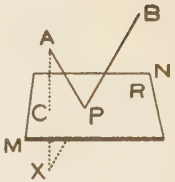
39. Construct a plane containing one given line and parallel to another.

40. Construct a plane through a given point parallel to any two given lines in space.

41. Construct a line through a given point in space which intersects two given lines not in the same plane.

When is there no such line? Is there ever more than one?

42. Find a point in a plane such that the sum of the two lines joining it to two fixed points on one side of the plane is the least possible.



Construction: Draw $AC \perp$ to plane MN and prolong it to X, making $CX = AC$. Draw BX, meeting plane MN at P. Draw AP. Take any other point R in plane MN.

Statement: $AP + PB < AR + RB$. Etc.

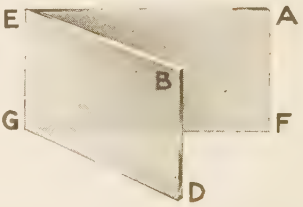
43. Find a point in a given plane equally distant from three given points. Is this ever impossible?

44. Find the one point equally distant from four given points not in the same plane.

Construction: Pass plane CM, containing points A, B, C, and plane CN, containing A, D, C. Find O, the center of the $\odot$ containing A, B, C. Find P, similarly. Draw the locus of points equally distant from A, B, C. (Consult 511.) Draw the locus of points equally distant from A, D, C. The plane $\perp$ to AC at its midpoint contains both these loci. (Explain.) Hence OX and PX intersect (?). $\therefore X$ is the required point.

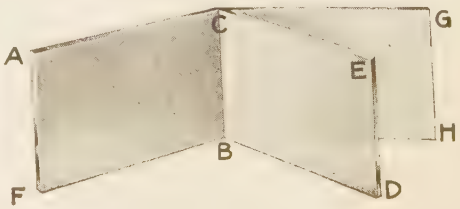
DIHEDRAL ANGLES

513. A **dihedral angle** is the amount of divergence of two intersecting planes. The **edge** of the dihedral angle is the line of intersection of the planes. The **faces** of the dihedral angle are the planes.



The intersecting planes AG and ED form the dihedral angle whose edge is EG, which is named A–GE–D; or, when there is only one dihedral angle at the edge, "the angle EG."

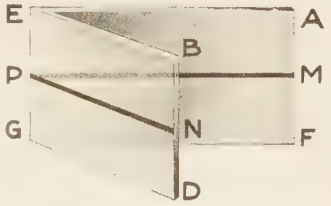
514. Adjacent dihedral angles are two dihedral angles that have the same edge and a common face between them.



Vertical dihedral angles are two dihedral angles that have the same edge, the faces of one being the extensions of the faces of the other.

515. The **plane angle** of a dihedral angle is the angle formed by two straight lines, one in each face, and perpendicular to the edge at the same point.

If PM is in plane AG and perpendicular to EG, and PN is in plane ED and perpendicular to EG at P, the angle MPN is the plane angle of the dihedral angle EG.



516. If one plane meets another, making the adjacent dihedral angles equal, these angles are **right dihedral angles.**

One plane is **perpendicular** to another plane if the dihedral angle formed by the two planes is a right dihedral angle.

517. Two dihedral angles are **equal** if they can be made to coincide.

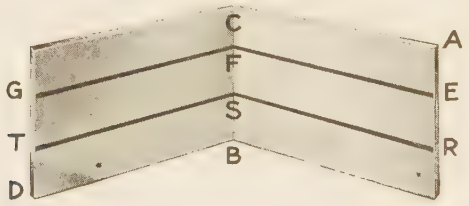
A dihedral angle is **acute**, **right**, or **obtuse** according as its plane angle is acute, right, or obtuse.

Dihedral angles are **complementary** or **supplementary**, **corresponding**, **alternate-interior**, etc., according as their plane angles are complementary or supplementary, corresponding, alternate-interior, etc.

NOTE. An open book often assists a student to a clear apprehension of the magnitude of dihedral angles. Thus he can see the angle increase during the act of opening the book, and observe the acute, right, and obtuse dihedrals. With the aid of two books he can understand better, perhaps, the meaning of complementary dihedrals, supplementary dihedrals, corresponding dihedrals, alternate-interior dihedrals, etc.

Proposition XXVII. Theorem

518. The plane angles of a dihedral angle are all equal.



Given: $\angle EFG$, the plane $\angle$ of dihedral $\angle BC$, at F, and $\angle RST$, the plane $\angle$ at S.

To Prove: $\angle EFG = \angle RST$.

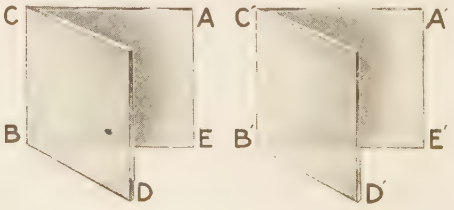
Proof:

	EF is $\parallel$ to RS	(62).
And	FG is $\parallel$ to ST	(?).
	$\therefore \angle EFG = \angle RST$ (499).	Q.E.D.

519. Corollary. The plane of the plane angle of a dihedral angle is perpendicular to the edge. (485).

Proposition XXVIII. Theorem

520. Two dihedral angles are equal if their plane angles are equal.



Given: Dihedral $\measuredangle CB$ and $C'B'$ whose plane $\measuredangle EBD$ and $E'B'D'$ are equal.

To Prove: Dih. $\angle CB$ = dih. $\angle C'B'$.

Proof:

	CB is $\perp$ to plane EBD.	
And	$C'B'$ is $\perp$ to plane $E'B'D'$	(519).

Apply dih. $\angle C'B'$ to dih. $\angle CB$ so that the plane $\angle E'B'D'$ coincides with its equal $\angle EBD$.

Now $C'B'$ coincides with CB (490).

$\therefore$ plane $C'D'$ coincides with plane CD and plane $C'E'$ coincides with plane CE (479).

$\therefore$ dih. $\angle CB =$ dih. $\angle C'B'$ (517).

Q.E.D.

Proposition XXIX. Theorem

521. If two dihedral angles are equal, their plane angles are equal. [Converse.]

Proof: Superpose dih. $\angle C'B'$ upon its equal dih. $\angle CB'$ making B' fall on B, and edge $B'C'$ on BC.

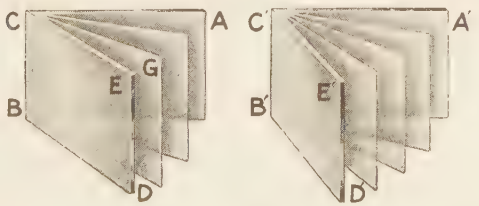
Then face $C'D'$ coincides with face CD, etc.

522. Corollary. Two vertical dihedral angles are equal. (See 520.)

523. Corollary. The plane angle of a right dihedral angle is a right angle; and if the plane angle of a dihedral angle is a right angle, the dihedral angle is right. (See 516.)

Proposition XXX. Theorem

524. Two dihedral angles have the same ratio as their plane angles.



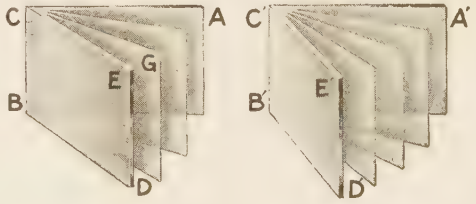
Given: Dihedral $\measuredangle A\text{–}BC\text{–}D$ and $A'\text{–}B'C'\text{–}D'$, having plane $\measuredangle ACE$ and $A'C'E'$, respectively.

To Prove:

Dih. $\angle A\text{–}BC\text{–}D$: dih. $\angle A'\text{–}B'C'\text{–}D' = \angle ACE : \angle A'C'E'$.

Proof: I. If the plane angles are **commensurable.**

There exists a common unit of measure of the plane $\measuredangle ACE$ and $A'C'E'$ (224).



Suppose this unit when applied to these angles is contained 3 times in $\angle ACE$ and 4 times in $\angle A'C'E'$.

$$\therefore \frac{\angle ACE}{\angle A'C'E'} = \frac{3}{4} \qquad \text{(Ax. 3).}$$

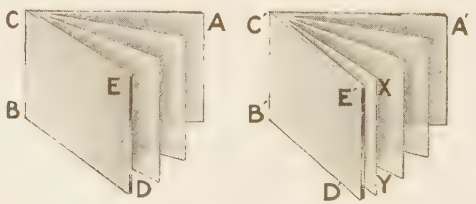
Pass planes through the edges and the several lines of division of the angles.

Dih. $\angle A\text{–}BC\text{–}D$ is divided into 3 parts; dih. $\angle A'\text{–}B'C'\text{–}D'$ is divided into 4 parts; all of these seven parts are equal (520).

$$\therefore \frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}D'} = \frac{3}{4} \qquad \text{(Ax. 3).}$$

$\therefore$ dih. $\angle A\text{–}BC\text{–}D$: dih. $\angle A'\text{–}B'C'\text{–}D' = \angle ACE : \angle A'C'E'$ (Ax. 1).

II. If the plane angles are **incommensurable.**



There does not exist a common unit. Suppose $\angle ACE$ to be divided into equal parts (any number of them).

Apply one of these as a unit of measure to $\angle A'C'E'$. There is a remainder, $XC'E'$, left over (because the $\measuredangle$ are incommensurable).

Pass a plane $C'Y$, determined by $B'C'$ and $C'X$.

Now $\frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}Y} = \frac{\angle ACE}{\angle A'C'X}$ (Commensurable ∡s).

Indefinitely increase the number of subdivisions of $\angle ACE$.

Then each part, that is, our unit or divisor, is indefinitely decreased. Hence $XC'E'$, the remainder, is indefinitely decreased.

That is, $\angle XC'E'$ approaches zero as a limit.

And dih. $\angle X\text{–}B'C'\text{–}D'$ approaches zero as a limit.

$\therefore \angle A'C'X$ approaches $A'C'E'$ as a limit; and dih. $\angle A'\text{–}B'C'\text{–}Y$ approaches dih. $\angle A'\text{–}B'C'\text{–}D'$ as a limit.

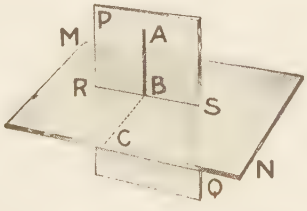
$\therefore \frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}Y}$ approaches $\frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}D'}$ as a limit;

and $\frac{\angle ACE}{\angle A'C'X}$ approaches $\frac{\angle ACE}{\angle A'C'E'}$ as a limit.

$\therefore \frac{\text{dih. } \angle A\text{–}BC\text{–}D}{\text{dih. } \angle A'\text{–}B'C'\text{–}D'} = \frac{\angle ACE}{\angle A'C'E'}$ (229). Q.E.D.

Proposition XXXI. Theorem

525. If a straight line is perpendicular to a plane, any plane containing this line is perpendicular to the given plane.



Given: Line $AB \perp$ to plane MN; plane PQ containing AB and intersecting plane MN in RS.

To Prove: Plane PQ is $\perp$ to plane MN.

Proof: In plane MN draw $BC \perp$ to RS.

Now AB is $\perp$ to RS. (473).

$\therefore \angle ABC$ is the plane $\angle$ of dih. $\angle P\text{–}SR\text{–}N$. (515).

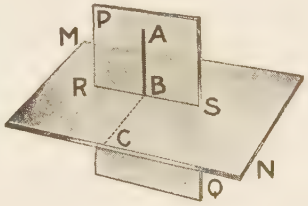
But $\angle ABC$ is a rt. $\angle$. (473).

$\therefore PQ$ is $\perp$ to MN (516). Q.E.D.

526. Corollary. If a plane is perpendicular to the edge of a dihedral angle, it is perpendicular to each face. (See 525.)

Proposition XXXII. Theorem

527. If one plane is perpendicular to another, any line in either plane, perpendicular to their intersection, is perpendicular to the other plane. [Converse of 525.]



Given: Plane $PQ \perp$ to plane MN; AB in plane $PQ \perp$ to the intersection, RS.

To Prove: $AB \perp$ to plane MN.

Proof: In plane MN draw $BC \perp$ to RS.

Now $\angle ABC$ is the plane angle of the dih. $\angle P\text{–}SR\text{–}N$. (515).

$\therefore \angle ABC$ is a rt. $\angle$ (523).

$\therefore AB$ is $\perp$ to BC (16).

But AB is $\perp$ to RS (Hyp.).

$\therefore AB$ is $\perp$ to plane MN (485).

Q.E.D.

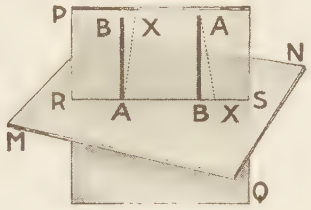
Ex. 1. In the figure of 527 prove BC perpendicular to the plane PQ. Also prove RS perpendicular to the plane ABC.

Ex. 2. In the figure of 527 prove plane ABC perpendicular to the planes MN and PQ.

Ex. 3. Under what condition will a line in one face of a dihedral angle meet a line in the other face?

Proposition XXXIII. Theorem

528. If one plane is perpendicular to another, a line drawn from any point in their intersection and perpendicular to one plane, lies in the other.



Given: Plane $PQ \perp$ to plane MN, intersecting in RS; AB (left-hand) $\perp$ to plane MN from A, in RS.

To Prove: AB is in plane PQ.

Proof: At A erect in plane PQ, $AX \perp$ to RS.

Then	AX is $\perp$ to plane MN	(527).
But	AB is $\perp$ to plane MN at A	(Hyp.).
	$\therefore$ AB and AX coincide	(490).
That is,	AB lies in plane PQ.	Q.E.D.

Proposition XXXIV. Theorem

529. If one plane is perpendicular to another, a line drawn from any point in one plane, and perpendicular to the other, lies in the first plane.

Given: Plane $PQ \perp$ to plane MN; AB (right-hand) $\perp$ to plane MN from A, any point in plane PQ.

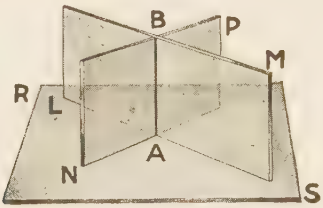
To Prove: AB lies in plane PQ.

Proof: From A draw in plane PQ, $AX \perp$ to RS.

Then	AX is $\perp$ to plane MN	(527).
	$\therefore$ AB and AX coincide	(491).
That is,	AB lies in plane PQ.	Q.E.D.

Proposition XXXV. Theorem

530. If two planes are perpendicular to a third plane, their intersection also is perpendicular to that plane.



Given: Planes *LM* and *NP*, each ⊥ to plane *RS*.

To Prove: The intersection *AB* is ⊥ to plane *RS*.

Proof: If, at *A*, a line is erected ⊥ to plane *RS*, it will lie in plane *LM* (528).

This ⊥ will lie also in plane *NP* (?).

∴ this ⊥ is the intersection *AB* (466).

That is, *AB* is ⊥ to plane *RS*. Q.E.D.

531. Corollary. If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection. (The same truth as 530.)

Ex. 1. If each of three planes is perpendicular to the other two, each of the three intersections is perpendicular to the remaining plane, and perpendicular to the other two intersections.

[Use figure of 530.]

Ex. 2. If two parallel planes are each perpendicular to a third plane, their intersections with that plane are parallel.

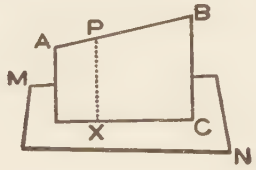
Ex. 3. Is Proposition XXXV true in the case of the intersecting walls of a building?

Ex. 4. What is the "exterior point" that, together with the plumb-bob, determines the vertical plane for the mason?

Ex. 5. If the builder keeps the edge of a building perpendicular to the plane of a level street, the two intersecting walls will also be perpendicular to the street. Why?

Proposition XXXVI. Theorem

532. Through a given line not perpendicular to a plane, one plane can be passed perpendicular to that plane, and only one.



Given: AB not $\perp$ to plane MN.

To Prove: Through AB one plane can be passed $\perp$ to MN, and only one.

Proof: I. From P, any point in AB, draw $PX \perp$ to MN.
Through AB and PX pass plane AC.
Plane AC is $\perp$ to plane MN (525).

II. Suppose another plane containing AB is $\perp$ to plane MN.
Then the intersection AB, of these two planes, which are $\perp$ to plane MN, will be $\perp$ to plane MN (530).
But AB is not $\perp$ to plane MN (Hyp.).
$\therefore$ there is only one plane containing AB that is $\perp$ to plane MN. Q.E.D.

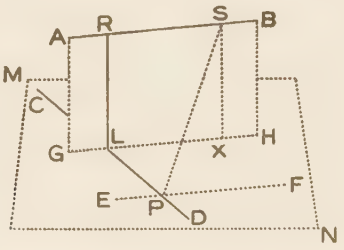
533. Corollary. The plane containing a straight line and its projection upon a plane is perpendicular to the given plane.

534. Corollary. If a line meets its projection on a plane, any line of the plane perpendicular to one of these lines at their intersection is perpendicular to the other also.

Proof: Use fig. of 505.
Plane ABP is $\perp$ to plane MN (533).
A line $\perp$ to plane ABP at B will lie in plane MN (528).
$\therefore$ this line is $\perp$ to both AB and PB (473).

Proposition XXXVII. Theorem

535. Between any two straight lines not in the same plane, one and only one common perpendicular can be drawn, and this common perpendicular is the shortest line that can be drawn between the two lines.



Given: Lines AB and CD not in the same plane.

To Prove: I. One line can be drawn $\perp$ to AB and CD.

II. Only one $\perp$ can be drawn.

III. This $\perp$ is the shortest line that can be drawn between AB and CD.

Proof: I. At P, any point in CD, draw $EF \parallel$ to AB.

Pass plane MN, containing CD and EF.

Pass plane AH through AB and $\perp$ to plane MN, intersecting plane MN in GH, and CD at L.

In plane AH draw $RL \perp$ to GH.

	Plane MN is $\parallel$ to AB	(482).
	GH is $\parallel$ to AB	(483).
	RL is $\perp$ to GH	(Const.).
	$\therefore$ RL is $\perp$ to plane MN	(527).
	$\therefore$ RL is $\perp$ to CD	(473).
Also	RL is $\perp$ to AB	(64).
That is,	RL is $\perp$ to both the given lines.	Q.E.D.

II. If another line can be drawn $\perp$ to AB and CD, suppose SP is this $\perp$. In plane AH draw $SX \perp$ to GH.

Then SX is $\perp$ to plane MN (527).

But if SP is $\perp$ to AB, it is $\perp$ to EF (64).

$\therefore$ SP is $\perp$ to plane MN (485).

Thus there are two $\perp$s from S to plane MN (SX and SP).

But this is impossible (491).

$\therefore$ there can be no second $\perp$ to these two given lines.

Q.E.D.

III. Suppose SP is any other line between AB and CD.

Now RL is $\parallel$ to SX (62).

$\therefore$ RX is a $\square$ (120).

$\therefore$ $RL = SX$ (124).

But $SX < SP$ (504, I).

$\therefore$ $RL < SP$ (Ax. 6).

That is, RL is shorter than any other line between AB and CD. Q.E.D.

Ex. 1. In the figure of 535 prove that a plane perpendicular to RL at its midpoint will be parallel to AB and CD.

Ex. 2. Prove, also, that this plane will bisect SP.

Ex. 3. Prove that if CD is not perpendicular to EF, no plane can be passed through AB, perpendicular to CD.

Ex. 4. Tell how we can construct one plane perpendicular to another.

Ex. 5. Tell how we can construct one plane through a given point, perpendicular to any two given planes.

Ex. 6. Tell how we can construct a plane containing a given line and perpendicular to a given plane.

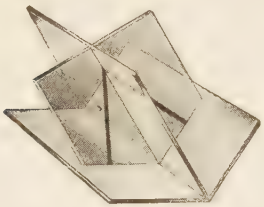
Ex. 7. Tell, so that a blind boy could understand, how to draw a line perpendicular to any two lines in space (not in the same plane).

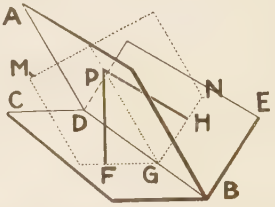
Ex. 8. If two planes are parallel, what is the form of the projection on one plane, of a circle in the other?

Ex. 9. If two planes are perpendicular, what is the form of the projection on one plane, of a circle in the other?

Proposition XXXVIII. Theorem

536. Every point in a plane bisecting a dihedral angle is equally distant from the faces of the angle.





Given: Plane AB, bisecting the dih. $\angle C\text{–}BD\text{–}E$; any point P in plane AB; $PF \perp$ to face CB; $PH \perp$ to face DE.

To Prove: $PF = PH$.

Proof: Pass plane MN, containing PF and PH, intersecting CB in FG, AB in PG, DE in HG, BD at G.

Now plane MN is $\perp$ to planes CB and DE (525).

$\therefore$ plane MN is $\perp$ to BD (531).

$\therefore$ BG is $\perp$ to FG, PG, and HG (473).

Hence $\angle PGF$ is the plane $\angle$ of dih. $\angle A\text{–}BD\text{–}C$ and $\angle PGH$ is the plane $\angle$ of dih. $\angle A\text{–}BD\text{–}E$ (515).

These dih. $\measuredangle$ are $=$ (Hyp.).

$\therefore$ $\angle PGF = \angle PGH$ (521).

$\measuredangle PFG$ and PHG are rt. $\measuredangle$ (473).

In the right $\triangle$ PFG and PGH, $PG = PG$ (?).

$\angle PGF = \angle PGH$. $\therefore$ $\triangle PFG \cong \triangle PGH$ (?).

$\therefore$ $PF = PH$ (?). Q.E.D.

537. Corollary. **Any point in a dihedral angle and equally distant from its faces is in the plane bisecting the angle.**

To Prove: The plane AB, determined by the point P and the edge BD bisects the dih. $\angle C\text{–}BD\text{–}E$.

538. Corollary. **The locus of points within a dihedral angle and equally distant from its faces is the plane bisecting that angle.** (Proof: 536, 537.)

ORIGINAL EXERCISES

1. Are two planes perpendicular to the same plane necessarily parallel?

2. A straight line and a plane perpendicular to the same plane are parallel.

3. A plane perpendicular to a line in another plane is perpendicular to that plane.

4. If three planes, all perpendicular to a fourth, intersect in three lines, these lines are parallel, in pairs.

5. If the projection of any line (straight or curved) upon a plane is a straight line, the line is entirely in one plane.

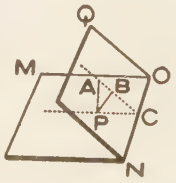
6. The angle between the normals drawn to the faces of a dihedral angle from a point within the angle is the supplement of the plane angle of the dihedral angle.

7. If a line is parallel to a plane, any plane perpendicular to the line is perpendicular also to the plane.

[Construct the projection of the given line upon the given plane.]

8. What is the locus of points in space equally distant from two intersecting planes?

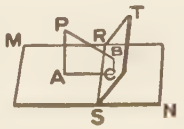
9. If from any point in a face of a dihedral angle, a normal is drawn to each face, the plane of these normals is perpendicular to the edge of the dihedral angle.



10. If from any point in a face of a dihedral angle, a normal is drawn to each face, the angle they form is equal to the plane angle of the dihedral angle.

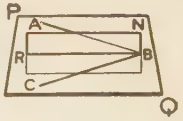
11. If a line is perpendicular to a plane, any plane parallel to the line is also perpendicular to the plane.

12. If PA is a normal to plane MN, PB a normal to plane ST, and BC a normal to plane MN, AC is perpendicular to RS, the intersection of planes MN and ST.



13. The plane perpendicular to the line that is perpendicular to two lines in space, at its middle point, bisects every straight line having its extremities in these lines.

14. The plane perpendicular to the plane of an angle and containing the bisector of the angle is the locus of points equally distant from the sides of the angle.



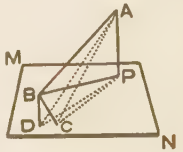
Proof: The ⊥s from any point in plane NR to AB and BC will have equal projections. (Explain by use of 94.)

$\therefore$ these perpendiculars are equal (?).

15. What is the locus of points in space equally distant from two intersecting lines?

16. If AP is a normal to plane MN and if angle PBC, in plane MN, is a right angle, angle ABC also is a right angle.

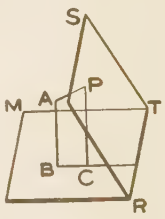
[Prove BC is ⊥ to plane APB.]



17. If AP is a normal to plane MN, and $\angle PBD$, in plane MN, is obtuse, $\angle ABD$ also is obtuse.

Proof: Take $BC = BD$; prove $PD > PC$. Then prove $AD > AC$, etc.

18. PA is perpendicular to plane RS; AB and PC are perpendicular to plane MR. Prove BC perpendicular to RT.



19. If two parallel planes are cut by a third plane, the alternate-interior dihedral angles are equal; the corresponding dihedral angles are equal; the alternate-exterior dihedral angles are equal; the adjoining interior dihedral angles are supplementary.

20. State and prove the converse theorems of those in No. 19.

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21. Construct a plane perpendicular to a given plane and containing a given line in that plane.

22. Construct a plane perpendicular to a given plane and containing a given line without that plane.

23. Construct through a given point a line which will intersect any two given lines not in the same plane.

Construction: Pass a plane through the point and one of the lines. This plane intersects the other line at a point, etc.

24. To bisect a given dihedral angle.

Construction: Pass a plane ⊥ to the edge. Bisect the plane ∠ of the given dihedral, etc.

25. Construct a line each of whose points shall be equally distant from the ends of a given line and also equally distant from the faces of a dihedral angle.

26. Find the locus of points equally distant from two points and equally distant from two intersecting planes. Discuss.

27. Find a point equally distant from three given points and equally distant from two intersecting planes. Is this problem ever impossible? Will there ever be two points? When will there be only one point?

28. Find a point equally distant from three given points and equally distant from two intersecting lines. Discuss fully.

POLYHEDRAL ANGLES

539. If three or more planes meet at a point, they form a **polyhedral angle.** The opening partially surrounded by the planes is the polyhedral angle.

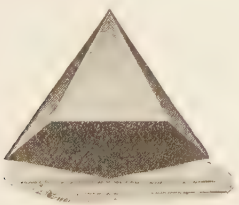
The point common to all the planes is the **vertex.**

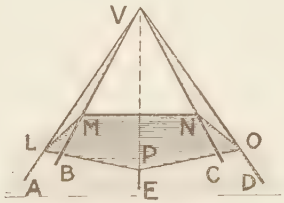
The planes are the **faces.**

The intersections of adjacent faces are the **edges.**

The angles formed at the vertex, by adjacent edges, are the **face angles.**

Thus, $V - ABCDE$ is a polyhedral angle; V is the vertex; AV, BV, etc., are edges; planes AVB, BVC, etc., are faces; $\angle AVB$, BVC, etc., are face angles.

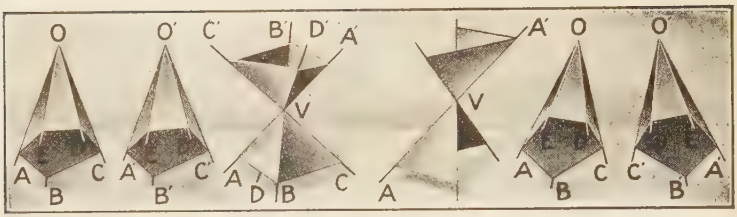




540. A **plane section** of a polyhedral angle is the plane figure bounded by the intersections of all the faces by a plane.

Polygon $LMNOP$ is a plane section of polyhedral angle $V - ABCDE$.

A **convex polyhedral angle** is one whose plane sections are all convex.



EQUAL POLYHEDRAL ANGLES | VERTICAL POLYHEDRAL ANGLES | VERTICAL DIHEDRAL ANGLES | SYMMETRICAL POLYHEDRAL ANGLES

541. Two polyhedral angles are **equal** if they can be made to coincide in all particulars. That is, if two polyhedral angles are equal, their homologous dihedral angles are equal; their homologous face angles are equal, and they are arranged in the same order. The length of the edges or the extent of the faces does not affect the size of the angle.

Two polyhedral angles are **vertical** if the edges of one are the prolongations of the edges of the other.

Two polyhedral angles are **symmetrical** if all the parts of one are equal to the corresponding parts of the other, but arranged in opposite order.

NOTE. It is apparent from the definitions that equal polyhedral angles are mutually equiangular as to the face angles and as to the dihedral angles.

Vertical polyhedral angles are mutually equiangular as to their face angles and as to their dihedral angles, but the order is reversed.

Symmetrical polyhedral angles are also mutually equiangular as to their face angles and as to their dihedral angles, but the order is reversed. Thus, if one follows around the polygon $A'D'$ in alphabetical order, he is moving as the hands of a clock, if the eye is at the vertex O'; but if he follows around AD alphabetically, he is moving in a direction opposite to the motion of the hands of a clock, if the eye is at the vertex O. Hence, it is apparent that, in general, symmetrical polyhedral angles cannot be made to coincide.

542. A **trihedral angle** is a polyhedral angle having three and only three faces.

A trihedral angle is **rectangular** if it contains a right dihedral angle ; **birectangular** if it contains two right dihedral angles; **trirectangular** if it contains three right dihedral angles.

A trihedral angle is **isosceles** if two of its face angles are equal.

PRELIMINARY THEOREMS

543. Two vertical polyhedral angles are symmetrical.

Proof : Their homologous face angles are equal and arranged in reverse order, and their homologous dihedral angles are equal and arranged in reverse order.

$\therefore$ they are symmetrical (Def. 541).

544. If two polyhedral angles are symmetrical, the vertical polyhedral angle of the one is equal to the other.

Because the corresponding parts are equal and they are arranged in the same order.

545. Provided two trihedral angles have their parts arranged in the same order, they are equal :

I. If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

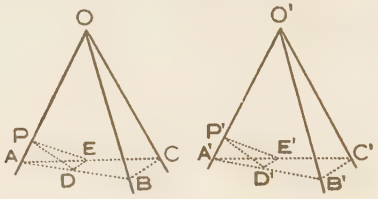
II. If a face angle and the two dihedral angles adjoining it of the one are equal respectively to a face angle and the two dihedral angles adjoining it, of the other.

Proof : By method of superposition, as in plane $\triangle$.

Ex. Illustrate a birectangular trihedral angle by means of an open book standing on a table. Similarly illustrate a trirectangular trihedral angle. Similarly use two closed books to illustrate a rectangular trihedral angle.

Proposition XXXIX. Theorem

546. Provided two trihedral angles have their parts arranged in the same order, they are equal, if the three face angles of one are equal respectively to the three face angles of the other.



Given: Trih. $\measuredangle$ O and O';

$$\angle AOB = \angle A'O'B';$$
$$\angle BOC = \angle B'O'C';$$
$$\angle COA = \angle C'O'A'.$$

To Prove: Trih. $\angle O$ = trih. $\angle O'$, that is, dih. $\angle OA$ = dih. $\angle O'A'$, etc.

Proof: Take $OA = OB = OC = O'A' = O'B' = O'C'$.

Draw AB, BC, AC, $A'B'$, $B'C'$, $A'C'$.

Take, on edges AO and $A'O'$, $AP = A'P'$ and in face AOB draw $PD \perp$ to AO.

$\angle OAB$ is acute ($\triangle AOB$ is isosceles). $\therefore$ PD will meet AB.

In face AOC draw $PE \perp$ to AO, meeting AC at E.

Draw DE. Similarly draw $P'D'$, $P'E'$, $D'E'$.

Now $\angle EPD$ and $E'P'D'$ are the plane $\measuredangle$ of the dihedral $\measuredangle$ AO and $A'O'$ (515).

To prove that these $\measuredangle$ are equal requires the proof that eight pairs of $\triangle$ are equal.

(1) $\triangle OAB = \triangle O'A'B'$ (Explain).
(2) $\triangle OBC = \triangle O'B'C'$ (Explain).
(3) $\triangle OAC = \triangle O'A'C'$ (Explain).

$\left.\begin{array}{r} \therefore AB = A'B' \\ BC = B'C' \\ AC = A'C' \end{array}\right\}$ (?).

$\angle OAB = \angle O'A'B'$,
$\angle OAC = \angle O'A'C'$ (?)

$$\left.\begin{array}{l}(4)\ \triangle APD = \triangle A'P'D' \quad \text{(Explain).}\\ (5)\ \triangle APE = \triangle A'P'E' \quad \text{(Explain).}\end{array}\right\} \left.\begin{array}{c}\therefore\ AD = A'D'\\ AE = A'E'\\ PD = P'D', \text{ etc.}\end{array}\right\} (?).$$

$$(6)\ \triangle ABC = \triangle A'B'C' \quad \text{(Explain).} \quad \therefore \angle CAB = \angle C'A'B' \quad (?).$$

$$(7)\ \triangle AED = \triangle A'E'D' \quad \text{(Explain).} \quad \therefore ED = E'D' \quad (?).$$

$$(8)\ \triangle PED = \triangle P'E'D' \quad \text{(Explain).} \quad \therefore \angle EPD = \angle E'P'D' \quad (?).$$

Hence dih. $\angle AO$ = dih. $\angle A'O'$ (520).

Similarly, one may prove the other pairs of homologous dihedral angles equal.

$\therefore$ trihedral $\angle O$ = trihedral $\angle O'$ (541).

Q.E.D.

Proposition XL. Theorem

547. **Provided two trihedral angles have their parts arranged in reverse order, they are symmetrical:**

I. If two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other.

II. If a face angle and the two dihedral angles adjoining it of the one are equal respectively to a face angle and the two dihedral angles adjoining it of the other.

III. If the three face angles of one are equal respectively to the three face angles of the other.

Proof: In each case construct a third trihedral $\angle$ symmetrical to the first. This third figure will have its parts = to the parts of the second, and arranged in the same order (Def. 541).

$\therefore$ the third = the second (545 and 546).

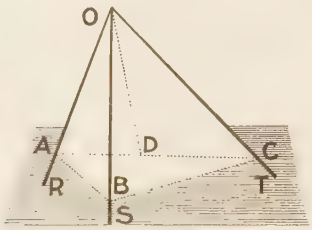
$\therefore$ the first is symmetrical to the second. (Ax. 6.)

Q.E.D.

Ex. The three planes bisecting the three dihedral angles of a trihedral angle intersect in a straight line.

Proposition XLI. Theorem

548. The sum of any two face angles of a trihedral angle is greater than the third face angle.



Given: Trih. $\angle O\text{-}RST$ in which face angle ROT is the greatest.

To Prove: $\angle ROS + \angle SOT > \angle ROT$.

Proof: Construct, in face ROT, $\angle ROD = \angle ROS$.

Take $OD = OB$; draw ADC, meeting OT at C. Draw AB and BC.

	$\triangle AOD \cong \triangle AOB$	(Explain).
	$\therefore AB = AD$	(?).
Now	$AB + BC > AD + DC$	(Ax. 12)
But	$AB = AD$	(?).
Subtracting,	$BC > DC$	(Ax. 7).

Now $OB = OD$ (?), $OC = OC$ (?) and $BC > DC$ (Just proved).

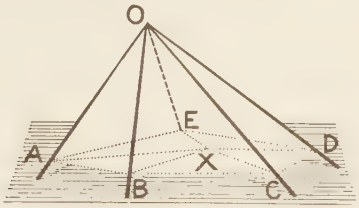
	$\therefore \angle BOC > \angle DOC$	(92).
But	$\angle AOB = \angle AOD$	(?).
Adding,	$\angle AOB + \angle BOC > \angle AOC$	(Ax. 7).
That is,	$\angle ROS + \angle SOT > \angle ROT$	(Ax. 6).

Ex. 1. Prove theorem of 548 for the case of an isosceles trihedral angle.

Ex. 2. The three planes containing the three bisectors of the three face angles of a trihedral angle and perpendicular to those faces intersect in a straight line.

Proposition XLII. Theorem

549. The sum of the face angles of any polyhedral angle is less than four right angles or 360°.



Given: Polyhedral $\angle O$, having n faces.

To Prove: The sum of the face $\measuredangle$ at $O < 4$ rt. $\measuredangle$ or 360°.

Proof: Pass a plane AD, intersecting all the faces, and the edges at A, B, C, etc.

In this section take any point X and join X to all the vertices of the polygon.

(1) There are n face $\triangle$ having their vertices at O (Hyp.).

(2) There are n base $\triangle$ having their vertices at X (Const.).

(3) The sum of the $\measuredangle$ of the face $\triangle = 2n$ rt. $\angle$ (104).

(4) The sum of the $\measuredangle$ of the base $\triangle = 2n$ rt. $\angle$ (104).

(5) $\therefore$ the sum of the $\measuredangle$ of the face $\triangle$ = the sum of the $\measuredangle$ of the base $\triangle$ (Ax. 1).

Now $\angle OAE + \angle OAB > \angle EAB$ (548).

And $\angle OBA + \angle OBC > \angle ABC$ (?), etc., etc.

Adding, the sum of the base $\measuredangle$ of the face $\triangle >$ the sum of the base $\measuredangle$ of the base $\triangle$ (Ax. 8).

Subtracting this inequality from equation (5) above, the sum of the face $\measuredangle$ at $O <$ the sum of the $\measuredangle$ at X (Ax. 9).

But the sum of all the $\measuredangle$ at $X = 4$ rt. $\measuredangle$ (47).

$\therefore$ the sum of the face $\measuredangle$ at $O < 4$ rt. $\measuredangle$, or 360° (Ax. 6).

Q.E.D.

ORIGINAL EXERCISES

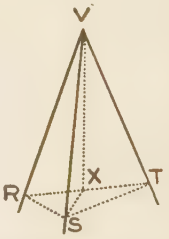
1. In the figure of 549, as the vertex O approaches the base, does the sum of the face angles at O increase or decrease? What limit does this sum approach? Does the sum ever become equal to this limit?

2. Can a polyhedral angle have for its faces three equilateral triangles? four? five? six? seven?

3. Can a polyhedral angle have for its faces four squares? five? three?

4. What other regular polygons can be used for the faces of a polyhedral angle?

5. If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.



Given: $\angle RVS = \angle SVT$.

To Prove: Dih. $\angle RV$ = dih. $\angle TV$.

Proof: Pass plane SVX bisecting dih. $\angle SV$. Prove trih. $\trianglelefteq V\text{–}RSX$ and $V\text{–}TSX$ are sym. by (547, 1).

6. An isosceles trihedral angle and its symmetrical trihedral angle are equal.

7. Find the locus of points equally distant from the three faces of a trihedral angle.

8. Find the locus of points equally distant from the three edges of a trihedral angle.

9. If the three face angles of a trihedral angle are equal, the three dihedral angles also are equal.

10. If the three face angles of a trihedral angle are right angles, the three dihedral angles also are right angles.

11. In any trihedral angle the greatest dihedral angle has the greatest face angle opposite it.

12. If the edges of one trihedral angle are perpendicular to the faces of a second trihedral angle, then the edges of the second are perpendicular to the faces of the first.

13. Construct, through a given point, a plane which shall make, with the faces of a polyhedral angle having four faces, a section that is a parallelogram.

Construction: Extend one pair of opp. faces to obtain their line of intersection. Similarly extend the other pair. Any plane section ∥ to these lines will be a $\square$. [Explain.]

BOOK VII

POLYHEDRONS

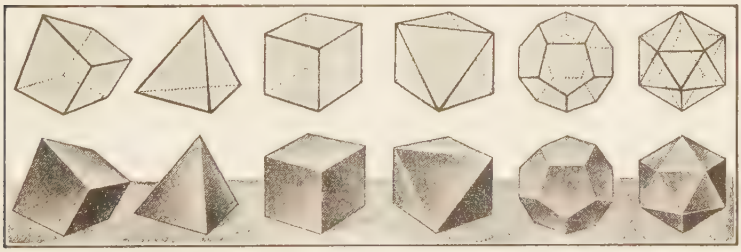
550. A **polyhedron** is a solid bounded by planes.

The **edges** of a polyhedron are the intersections of the bounding planes.

The **faces** are the portions of the bounding planes included by the edges.

The **vertices** are the intersections of the edges.

The **diagonal** of a polyhedron is a straight line joining two vertices not in the same face.



POLY-HEDRON | TETRA-HEDRON | HEXAHEDRON CUBE | OCTA-HEDRON | DODECA-HEDRON | ICOSA-HEDRON

551. A **tetrahedron** is a polyhedron having four faces.

A **hexahedron** is a polyhedron having six faces.

An **octahedron** is a polyhedron having eight faces.

A **dodecahedron** is a polyhedron having twelve faces.

An **icosahedron** is a polyhedron having twenty faces.

552. A polyhedron is **convex** if the section made by every plane is a convex polygon.

Only convex polyhedrons are considered in this book.

PRISMS

553. A **prism** is a polyhedron two of whose opposite faces are congruent polygons in parallel planes, and whose other faces are all parallelograms.

The **bases** of a prism are the congruent parallel polygons.

The **lateral faces** of a prism are the parallelograms.

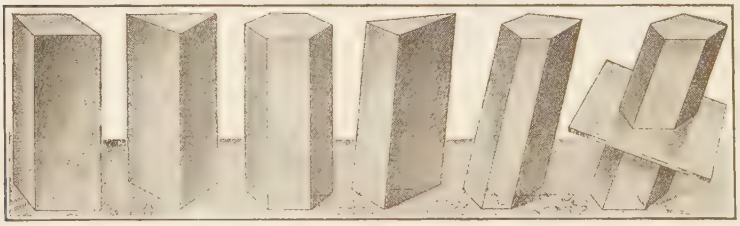
The **lateral edges** of a prism are the intersections of the lateral faces.

The **lateral area** of a prism is the sum of the areas of the lateral faces.

The **total area** of a prism is the sum of the lateral area and the areas of the bases.

The **altitude** of a prism is the perpendicular distance between the planes of the bases.

A **triangular prism** is a prism whose bases are triangles.



PRISM TRIANGULAR PRISM REGULAR PRISM OBLIQUE PRISMS RIGHT SECTION TRUNCATED PRISMS

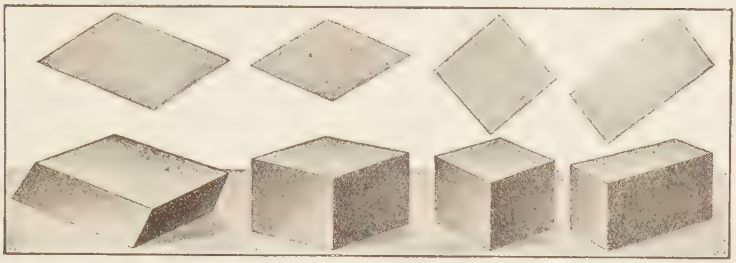
554. A **right prism** is a prism whose lateral edges are perpendicular to the planes of the bases.

A **regular prism** is a right prism whose bases are regular polygons.

An **oblique prism** is a prism whose lateral edges are not perpendicular to the planes of the bases.

A **truncated prism** is the portion of a prism included between the base and a plane not parallel to the base.

A **right section** of a prism is the section made by a plane perpendicular to the lateral edges of the prism.



PARALLELEPIPED | RIGHT PARALLELEPIPED | CUBE | RECTANGULAR PARALLELEPIPED

555. A **parallelepiped** is a prism whose bases are parallelograms.

A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to the planes of the bases.

A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

An **oblique parallelepiped** is a parallelepiped whose lateral edges are not perpendicular to the planes of the bases.

A **cube** is a rectangular parallelepiped whose six faces are squares.

556. The **unit of volume** is a cube whose edges are each a unit of length. The **volume of a solid** is the number of units of volume it contains. The volume of a solid is the ratio of that solid to the unit of volume.

The three edges of a rectangular parallelepiped meeting at any vertex are the **dimensions** of the parallelepiped.

Equal solids are solids that have equal volumes.

Congruent solids are solids that can be made to coincide.

Ex. What is the base of a rectangular parallelepiped? of a right parallelepiped? of an oblique parallelepiped? Illustrate, by removing the cover and bottom of an ordinary cardboard box and distorting the shape of the frame that remains, the three kinds of parallelepipeds.

NOTE. The space that is bounded by the surfaces of a solid, independent of the solid, is called a **geometrical solid.**

That is, if a material or physical body should occupy a certain position and then be removed elsewhere, there is a definite portion of space that is the same shape and size as the solid, and can be conceived as bounded by exactly the same surfaces as bounded the solid when in that original position. In order that we may pass planes and draw lines through solids, and superpose one solid upon another, it is convenient in studying the properties of solids to consider them usually as geometric solids, — the material body being removed for the time.

PRELIMINARY THEOREMS

557. THEOREM. **The lateral edges of a prism are equal.** (?.)

558. THEOREM. **Any two lateral edges of a prism are parallel.** (495.)

559. THEOREM. **Any lateral edge of a right prism equals the altitude.** (509.)

560. THEOREM. **The lateral faces of a right prism are perpendicular to the bases.** (525.)

561. THEOREM. **The lateral faces of a right prism are rectangles.** (Def. 554.)

562. THEOREM. **The faces and bases of a rectangular parallelepiped are rectangles.** (Def. 555.)

563. THEOREM. **All the faces of any parallelepiped are parallelograms.** (?)

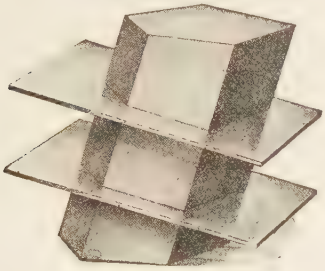
564. AXIOM. **A polyhedron cannot have fewer than four faces.**

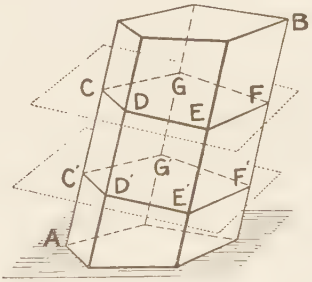
565. AXIOM. **A polyhedron cannot have fewer than three faces at each vertex.**

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

566. The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.





Given: Prism AB; ∥ sections CF and $C'F'$.

To Prove: Polygon $CF \cong$ polygon $C'F'$.

Proof:

CD is ∥ to $C'D'$, DE is ∥ to $D'E'$, etc.	(484).
$\therefore CD'$, DE', EF', etc. are [S]	(?).
$\therefore CD = C'D'$, $DE = D'E'$, $EF = E'F'$, etc.	(124).
$\angle GCD = \angle G'C'D'$, $\angle CDE = \angle C'D'E'$, etc.	(499).
$\therefore$ Polygon $CF \cong$ polygon $C'F'$	(150).

Q.E.D.

Ex. 1. All right sections of a prism are equal.

Ex. 2. Any section of a parallelepiped made by a plane cutting two pairs of opposite faces is a parallelogram.

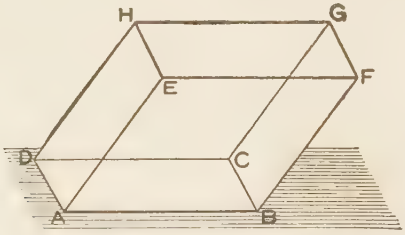
Ex. 3. How many edges has a cube? how many vertices? how many dihedral angles? how many trihedral angles?

Ex. 4. In the figure of 566, if DE is the altitude of one of the lateral faces, how could one express the area of that face?

Ex. 5. In 566, is CF' a prism? Why? Is CB a prism? Why? Is AF a prism? Why? What name is given CB?

Proposition II. Theorem

567. The opposite faces of a parallelepiped are congruent and parallel.



Given: (?).

To Prove: Face $AF \cong$ and $\parallel$ to face DG.

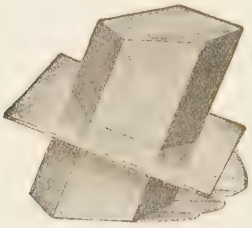
Proof:

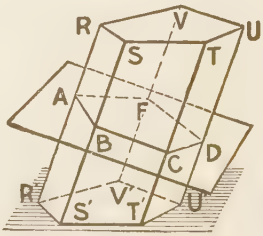
Faces AF and DG are $\square^{S}$	(?).
$AB = DC$, $AE = DH$	(124).
AB is $\parallel$ to DC and AE is $\parallel$ to DH	(?).
$\therefore \angle EAB = \angle HDC$	(499).
$\therefore$ face $AF \cong$ face DG	(133).
Also face AF is $\parallel$ to face DG	(499).

Q.E.D.

Proposition III. Theorem

568. The lateral area of a prism is equal to the product of a lateral edge by the perimeter of a right section.





Given: Prism RU'; edge $= E$; right section AD.

To Prove: Lateral area of $RU' = E \times$ perimeter of AD.

Proof: AB is $\perp$ to RR', BC is $\perp$ to SS', etc. (473).

Area $\square\ RS' = E \cdot AB$ (359).

Area $\square\ ST' = E \cdot BC$ (?).

Area $\square\ TU' = E \cdot CD$ (?).

etc. etc. Adding,

The lateral area $= E \cdot (AB + BC + CD +$ etc.) (Ax. 2).

$= E \cdot$ perimeter of rt. sect. (Ax. 6).

Q.E.D.

569. Corollary. **The lateral area of a right prism is equal to the product of its altitude and the perimeter of its base.**

$$L = H \cdot Pr.$$

(Where L = lateral area, H = altitude, and Pr = perimeter of base, of a right prism.)

570. Corollary. $T = L + 2B$.

(Where T = total area, and B = area of base.)

Ex. 1. Any section of a parallelepiped made by a plane parallel to any edge is a parallelogram.

Ex. 2. The sum of the face angles at all the vertices of any parallelepiped is equal to 24 right angles.

Ex. 3. The sum of the plane angles of all the dihedral angles of any parallelepiped is equal to 12 right angles.

Proof: Pass three planes $\perp$ to three intersecting edges. Prove these sections $\square$ whose $\measuredangle$ are the plane angles of the dihedral angles, etc.

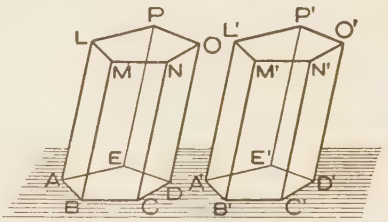
Ex. 4. Find the lateral area of a right prism whose altitude is 8 ft. and each side of whose triangular base is 5 ft.

Ex. 5. Find the total area of a regular prism whose base is a regular hexagon, 10 in. on a side, if the altitude of the prism is 15 in.

Ex. 6. Find the lateral area of a prism whose edge is 12 in. and whose right section is a pentagon, the sides of which are 3 in., 5 in., 6 in., 9 in., and 11 in.

Proposition IV. Theorem

571. **Two prisms are congruent if three faces including a trihedral angle of one are congruent, respectively, to three faces including a trihedral angle of the other, and similarly placed.**



Given: Prisms AO and $A'O'$; face $AM \cong$ face $A'M'$; face $AP \cong$ face $A'P'$; face $AD \cong$ face $A'D'$.

To Prove:
Prism $AO \cong$ prism $A'O'$.

Proof: The three face $\measuredangle$ at A are respectively $=$ to the three face $\measuredangle$ at A' (27).

$\therefore$ trih. $\angle A =$ trih. $\angle A'$ (546).

Superpose prism AO upon prism $A'O'$, making the equal trihedral $\measuredangle A$ and A' coincide.

Face AD coincides with face $A'D'$, face AM with $A'M'$, face AP with $A'P'$ (They are $\cong$ by hyp.).

That is, point L falls on L'; M on M'; and P on P'.

$\therefore$ the plane LO falls upon the plane $L'O'$ (477).

Polygon $LO \cong$ polygon $L'O'$ (Ax. 1).

$\therefore$ these bases coincide (?).

Similarly face BN coincides with $B'N'$, CO with $C'O'$, etc.

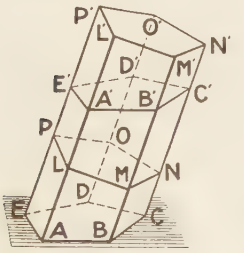
$\therefore$ the prisms are $\cong$ (Def. 556). Q.E.D.

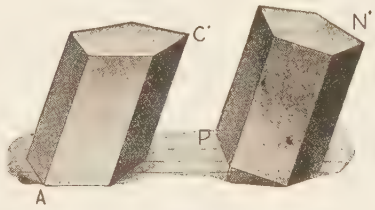
572. Corollary. **Two right prisms are congruent if they have congruent bases and equal altitudes.** (Explain.)

573. Corollary. **Two truncated prisms are congruent if three faces including a trihedral angle of one are congruent, respectively, to three faces including a trihedral angle of the other, and are similarly placed.** (Explain.)

Proposition V. Theorem

574. An oblique prism is equal to a right prism whose base is a right section of the oblique prism, and whose altitude is equal to the lateral edge of the oblique prism.





Given: Oblique prism AC'; right prism PN' whose base is PN, a right section of AC', and whose altitude $PP' =$ edge EE'.

To Prove: Oblique prism $AC' =$ right prism PN'.

Proof:

Edge $EE' = PP'$ (Hyp.).

Subtract PE' from each, and $EP = E'P'$ (Ax. 2).

Likewise $AL = A'L'$, $BM = B'M'$, $CN = C'N'$, etc.

(1) Face $AC \cong$ face $A'C'$ (553).

(2) In faces AP and $A'P'$, $EP = E'P'$, $AL = A'L'$ (Ax. 2).

Also $AE = A'E'$, $PL = P'L'$ (124).

That is, face AP and face $A'P'$ are mutually equilateral.

Also $\left.\begin{array}{l} \angle EAL = \angle E'A'L',\ \angle PEA = \angle P'E'A', \\ \angle ALP = \angle A'L'P',\ \angle EPL = \angle E'P'L' \end{array}\right\}$ (67).

That is, faces AP and $A'P'$ are mutually equiangular.

$\therefore$ face $AP \cong$ face $A'P'$ (150).

(3) Similarly, face $AM \cong$ face $A'M'$.

$\therefore$ truncated prism $AN \cong$ truncated prism $A'N'$ (573).

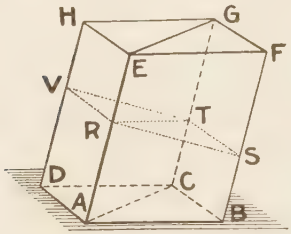
Now, add, solid $PC' =$ solid PC' (Iden.).

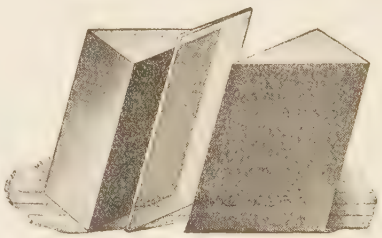
Oblique prism $AC' =$ right prism PN' (Ax. 2). Q.E.D.

Ex. Prove, in the figure of 574, that truncated prism AN is congruent to truncated prism $A'N'$, by the method of superposition.

Proposition VI. Theorem

575. The plane containing two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equal triangular prisms.





Given: Parallelepiped BH and plane AG containing the opposite edges AE and CG.

To Prove: Prism $ABC-F$ = prism $ADC-H$.

Proof: Pass a right section $RSTV$ intersecting the given plane in RT.

Face AF is $\parallel$ to face DG (?).

$\therefore$ RS is $\parallel$ to VT (484).

Also RV is $\parallel$ to ST (?).

$\therefore$ $RSTV$ is a $\square$ (?).

$\therefore$ $\triangle RST \cong \triangle RVT$ (126).

Prism ABC–F = a right prism whose base is RST and whose altitude = EA (574).

Prism ADC–H = a right prism whose base is RVT and whose altitude = EA (?).

But these imaginary right prisms are congruent (572).

$\therefore$ prism ABC–F = prism ADC–H (Ax. 1).

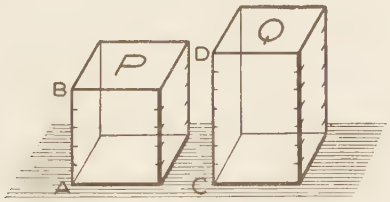
Q.E.D.

Ex. 1. The section of a parallelepiped made by a plane containing two diagonally opposite edges, is a parallelogram.

Ex. 2. Are the two triangular prisms in the diagram of 575 congruent? Why?

Proposition VII. Theorem

576. Two rectangular parallelepipeds having congruent bases are to each other as their altitudes.



Given: Rectangular parallelepipeds P and Q, having $\cong$ bases, and their altitudes AB and CD, respectively.

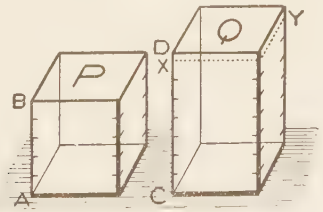
To Prove: $P : Q = AB : CD$.

Proof: I. If the altitudes are **commensurable**.

(Consult 524.)

II. If the altitudes are **incommensurable**.

There does not exist a common unit (225).



Suppose AB divided into equal parts. Apply one of these as a unit of measure to CD. There will be a remainder, DX (?).

Pass a plane XY, through X and $\parallel$ to the base.

Now $P : CY = AB : CX$ (?).

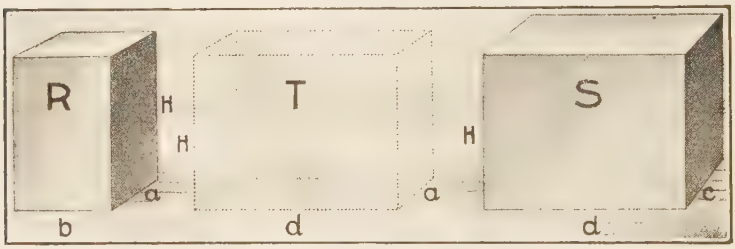
Indefinitely increase, etc. — as in 524.

577. Corollary. **Two rectangular parallelepipeds having two dimensions of the one equal respectively to two dimensions of the other, are to each other as their third dimension.**

The faces having the sides of one equal to the sides of the other, respectively, may be considered the bases and the third dimensions the altitudes. Thus this statement is the same as 576.

Proposition VIII. Theorem

578. Two rectangular parallelepipeds having equal altitudes are to each other as their bases.



Given: Rectangular parallelepipeds R and S, having the same altitude H; and other dimensions a, b, and c, d, respectively.

To Prove: $\frac{R}{S} = \frac{a \cdot b}{c \cdot d}$.

Proof: Construct a third rectangular parallelepiped, having altitude = to H, another dimension = to a, a third = to d.

$$\frac{R}{T} = \frac{b}{d} \text{ and } \frac{T}{S} = \frac{a}{c} \qquad (577).$$

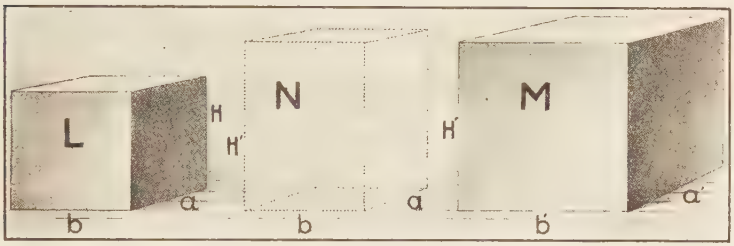
Multiplying, $$\frac{R}{S} = \frac{a \cdot b}{c \cdot d} \qquad (Ax.\ 3).$$

Q.E.D.

579. Corollary. **Two rectangular parallelepipeds having one dimension in common, are to each other as the products of the other dimensions.**

Proposition IX. Theorem

580. Theorem. **Any two rectangular parallelepipeds are to each other as the products of their three dimensions.**



Given: Rectangular parallelepipeds L and M, whose dimensions are a, b, H, and a', b', H', respectively.

To Prove: $\dfrac{L}{M} = \dfrac{a \cdot b \cdot H}{a' \cdot b' \cdot H'}$.

Proof: Construct N, whose dimensions are a, b, H'.

Then $\dfrac{L}{N} = \dfrac{H}{H'}$ (577).

And $\dfrac{N}{M} = \dfrac{a \cdot b}{a' \cdot b'}$ (579).

Multiplying, $\dfrac{L}{M} = \dfrac{a \cdot b \cdot H}{a' \cdot b' \cdot H'}$ (Ax. 3).

Q.E.D.

Ex. 1. In the above diagram, if $a = 6$ in., $b = 6$ in., and $H = 7$ in., find the length of the diagonal of L.

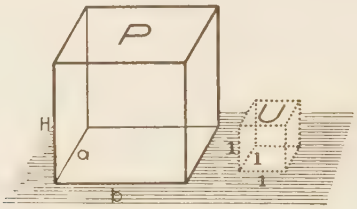
Ex. 2. Find the length of the diagonal of a room a ft. long, b ft. wide, and c ft. high.

Ex. 3. Show that the four diagonals of a rectangular parallelepiped are all equal.

Ex. 4. Find the diagonal of a cube whose edge is 4 in. Find the edge of a cube whose diagonal is 10 in.

Proposition X. Theorem

581. The volume of a rectangular parallelepiped is equal to the product of its three dimensions.



Given: (?). **To Prove:** (?).

Proof: Let U be a unit of vol.

$$\frac{P}{U} = \frac{a \cdot b \cdot H}{1 \cdot 1 \cdot 1} = a \cdot b \cdot H \qquad (580).$$

But $$\frac{P}{U} = \text{vol. of } P \qquad (556).$$

$$\therefore \text{ vol. of } P = a \cdot b \cdot H \qquad (\text{Ax. } 1).$$

Q.E.D.

582. Corollary. The volume of a rectangular parallelepiped is equal to the product of its base by its altitude.

$$V = B \cdot H. \qquad (\text{See } 581.)$$

583. Corollary. The volume of a cube is equal to the cube of its edge.

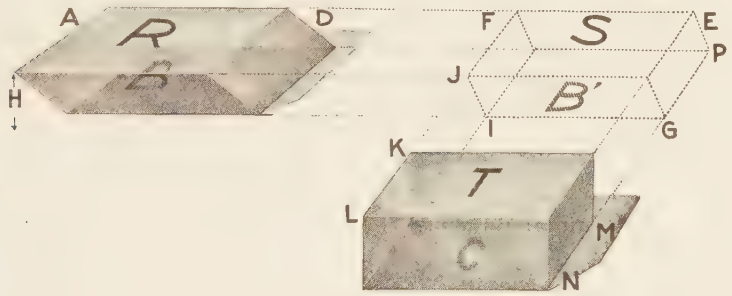
Ex. 1. A rectangular tank is 56 in. long, 44 in. wide, and 60 in. deep (inside). How many gallons will it hold? [231 cu. in. = 1 gal.]

Ex. 2. Three persons measured the above tank inaccurately. A found it to be 57 in. × 44 in. × 60 in.; B, 56 in. × 45 in. × 60 in.; and C, 56 in. × 44 in. × 61 in., — each recording two dimensions correctly, and one, 1 in. too great. Whose error was most serious, judging by the capacity of the tank? Which dimensions should be most carefully measured?

Ex. 3. What is the volume of a cube 6 in. on each edge? What is the edge of a cube having double the volume? What happens to the volume of a cube if we double each edge? if we halve each edge?

Proposition XI. Theorem

584. The volume of any parallelepiped is equal to the product of its base by its altitude.



Given: Parallelepiped R, whose base $= B$ and alt. $= H$.

To Prove: Volume of $R = B \cdot H$.

Proof: Prolong the edge AD and all edges $\parallel$ to AD.

On the prolongation of AD, take $EF =$ to AD.

Through E and F pass planes EG and FI, $\perp$ to EF, forming the right parallelepiped S.

Again, prolong FJ and all the edges $\parallel$ to FJ.

On the prolongation of FJ, take $KL =$ to FJ.

Through K and L pass planes KM and LN, $\perp$ to KL, forming the rectangular parallelepiped T.

Consider FI the base of S, and EF its altitude.

Then	$R = S$	(574).
Also	$B = B'$	(360).

Consider FP the base of S, KM the base of T, and KL its altitude.

Then	$S = T$; also $B' \cong C$	(574, 134).
Hence	$R = T$ and $B = C$	(Ax. 1).
And	the altitude of T is H	(509).
But	the volume of $T = C \cdot H$	(582).
	$\therefore$ V of $R = \mathrm{B} \cdot \mathrm{H}$ (Ax. 6).	Q.E.D.

585. Corollary. **Two parallelepipeds having equal altitudes and equal bases are equal.** (Ax. 1.)

586. Corollary. **Two parallelepipeds having equal altitudes are to each other as their bases.**

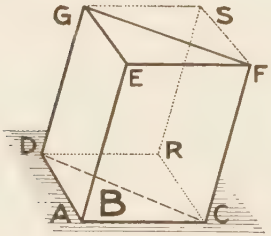
Proof: $Q = B \cdot H$ and $R = B' \cdot H$. $\therefore \frac{Q}{R} = \frac{B}{B'}$ (584, Ax. 3).

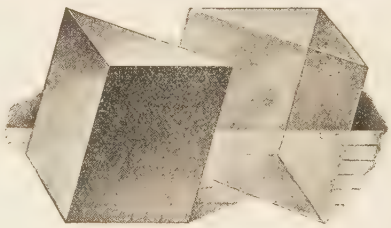
587. Corollary. **Two parallelepipeds having equal bases are to each other as their altitudes.** (?.)

588. Corollary. **Any two parallelepipeds are to each other as the products of their bases by their altitudes.** (?.)

Proposition XII. Theorem

589. The volume of a triangular prism is equal to the product of its base by its altitude.





Given: Triangular prism $ACD\text{–}F$; base $= B$; alt. $= H$.

To Prove: Volume of $ACD\text{–}F = B \cdot H$.

Proof: Construct parallelepiped AS having as three of its lateral edges AE, CF, DG.

Vol. $AS = ACRD \cdot H$ (584).

Hence $\frac{1}{2}$ volume of $AS = \frac{1}{2}\ ACRD \cdot H$ (Ax. 3).

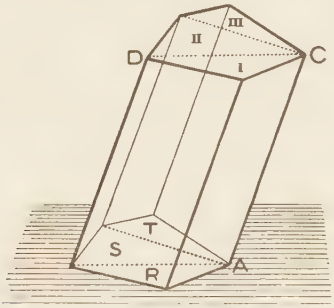
But $\frac{1}{2}$ volume of AS = volume of prism $ACD\text{–}F$ (575).

And $\frac{1}{2}\ ACRD = B$ (126).

$\therefore$ volume of $ACD\text{–}F = B \cdot H$ (Ax. 6). Q.E.D.

Proposition XIII. Theorem

590. The volume of any prism is equal to the product of its base by its altitude.



Given: Prism AD; base = to B; altitude = to H.

To Prove: Vol. of $AD = B \cdot H$.

Proof: Through any lateral edge, AC, and other lateral edges not adjoining AC, pass planes cutting the prism into triangular prisms I, II, III, having bases R, S, T, respectively.

Vol. of prism I $= R \cdot H$
Vol. of prism II $= S \cdot H$ (589).
Vol. of prism III $= T \cdot H$

Adding,

Vol. of prism $AD = (R + S + T)H = B \cdot H$ (Ax. 2). Q.E.D.

591. Corollary. Two prisms having equal altitudes and equal bases are equal.

592. Corollary. Two prisms having equal altitudes are to each other as their bases.

593. Corollary. Two prisms having equal bases are to each other as their altitudes.

594. Corollary. Any two prisms are to each other as the products of their bases by their altitudes.

ORIGINAL EXERCISES

1. Which rectangular parallelepiped contains the greater volume, one whose edges are 5 in., 7 in., 9 in., or one whose edges are 4 in., 6 in., 13 in.?

2. The base of a prism is a right triangle whose legs are 8 m. and 12 m. and the altitude of the prism is 20 m. Find its volume.

3. During a rain, half an inch of water fell. How many gallons fell on a level ten-acre park, allowing $7\frac{1}{2}$ gal. to the cubic foot?

4. Counting 38 cu. ft. of coal to a ton, how many tons will a coal bin 18 ft. long, 6 ft. wide, and $9\frac{1}{2}$ ft. deep contain, when even full?

5. How many faces has a parallelepiped? edges? vertices? How many faces has a hexagonal prism? edges? vertices?

6. Every lateral face of a prism is parallel to the lateral edges not in that face.

7. Every lateral edge of a prism is parallel to the faces that do not contain it.

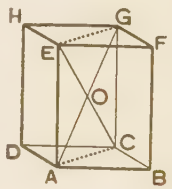
8. Every plane containing one and only one lateral edge of a prism is parallel to all the other lateral edges.

9. Any lateral face of a prism is less than the sum of the other lateral faces.

10. The diagonals of a rectangular parallelepiped are equal.

Proof: Pass the plane $ACGE$. This is a rectangle (?), etc.

11. The four diagonals of a parallelepiped bisect each other.

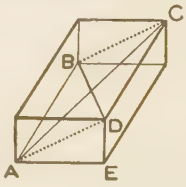


[First prove that one pair bisect each other; thus prove that any pair bisect each other, etc.]

12. Two triangular prisms are equal if their lateral faces are equal each to each.

13. Any prism is equal to the parallelepiped having the same altitude and an equal base.

14. The square of the diagonal of a rectangular parallelepiped is equal to the sum of the squares of its three dimensions.



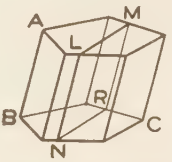
To Prove: $\overline{AC}^2 = \overline{AE}^2 + \overline{ED}^2 + \overline{DC}^2$.

Proof: AD is the hypotenuse of rt. $\triangle AED$, and AC, of rt. $\triangle ACD$.

15. The diagonal of a cube is equal to the edge multiplied by $\sqrt{3}$.

16. The volume of a triangular prism is equal to half the product of the area of any lateral face by the perpendicular drawn to that face from any point in the opposite edge.

17. Every section of a prism made by a plane parallel to a lateral edge is a parallelogram.



To Prove: $LMRN$ a $\square$. **Proof**: LM is $\parallel$ to NR (?). LN and MR are each $\parallel$ to any edge. (Explain.)

18. Every polyhedron has an even number of face angles.

Proof: Consider the faces as separate polygons. The number of sides of these polygons = double the number of edges of the polyhedron. (Explain.) But the number of sides of these polygons = the number of their angles, that is, the number of face angles. $\therefore$ the number of face angles = double the number of edges = an even number (?).

19. There is no polyhedron having fewer than 6 edges.

20. Find the contents and total area of a room 7 m. $\times$ 5 m. $\times$ 3 m.

21. Find the volume, lateral area, and total area of an 8-inch cube.

22. A right prism whose height is 12 ft. has for its base a right triangle whose legs are 6 ft. and 8 ft. Find the volume, lateral area, and total area of the prism.

23. Find the altitude of a rectangular parallelepiped whose base is 21 in. $\times$ 30 in., equivalent to a rectangular parallelepiped whose dimensions are 27 in. $\times$ 28 in. $\times$ 35 in.

24. A cube and a rectangular parallelepiped whose edges are 6 in., 16 in., and 18 in., have the same volumes. Find the edge of the cube.

25. Find the volume of a rectangular parallelepiped whose total area is 620 sq. in. and whose base is 14 in. $\times$ 9 in.

26. How many bricks each 8 in. $\times$ $2\frac{1}{4}$ in. $\times$ 2 in. will be required to build a wall 22 ft. $\times$ 3 ft. $\times$ 2 ft. (not allowing for mortar)?

27. If a triangular prism is 20 in. high and each side of its base is 8 in., how many cubic inches does it contain?

28. Find the lateral area, total area, and volume of a regular hexagonal prism each side of whose base is 10 in. and whose altitude is 15 in.

29. A box is 12 in. $\times$ 9 in. $\times$ 8 in. What is the length of its diagonal?

30. Each edge of a cube is 8 in. Find its diagonal.

31. The diagonal of a cube is $10\sqrt{3}$ in. Find its edge, volume, and total area.

32. A trench is 180 ft. long and 12 ft. deep, 7 ft. wide at the top and 4 ft. at the bottom. How many cubic yards of earth have been removed?

33. A metallic tank, open at the top, is made of iron 2 in. thick; the internal dimensions of the tank are, 4 ft. 8 in. long, 3 ft. 6 in. wide, 4 ft. 4 in. deep. Find the weight of the tank if empty; if full of water.

[Water weighs $62\frac{1}{2}$ lb. to the cubic foot and iron is 7.2 times as heavy as water.]

34. The base of a right parallelepiped is a rhombus whose sides are each 25 in., and the shorter diagonal is 14 in. The height of the parallelepiped is 40 in. Find its volume and total surface.

35. If the diagonal of a cube is 12 ft., find its surface.

36. If the total surface of a cube is 54 sq. ft., find its volume.

37. A right prism whose altitude is 25 in. has for its base a triangle whose sides are 11 in., 13 in., 20 in. Find its lateral area, total area, and volume.

PYRAMIDS

595. A **pyramid** is a polyhedron, one of whose faces is a polygon and whose other faces are all triangles having a common vertex.

The **lateral faces** of a pyramid are the triangles.

The **lateral edges** of a pyramid are the intersections of the lateral faces. The **vertex** of a pyramid is the common vertex of all the lateral faces.

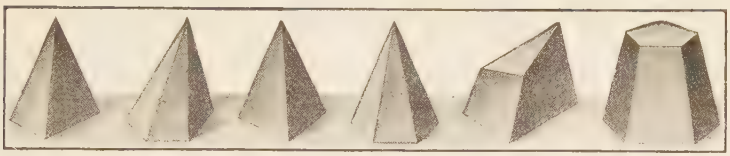
The **base** of a pyramid is the face opposite the vertex.

The **lateral area** of a pyramid is the sum of the areas of the lateral faces. The **total area** of a pyramid is the sum of the lateral area and the area of the base.

The **altitude** of a pyramid is the perpendicular distance from the vertex to the plane of the base.

A **triangular pyramid** is a pyramid whose base is a triangle. It is called also a tetrahedron. (See 551.)

596. A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude, from the vertex, meets the base at its center.



PYRAMIDS | REGULAR PYRAMIDS | TRUNCATED PYRAMID | FRUSTUM OF A PYRAMID

The **slant height** of a regular pyramid is the line drawn in a lateral face, from the vertex perpendicular to the base of the triangular face. It is the altitude of any lateral face.

597. The **frustum** of a pyramid is the part of a pyramid included between the base and a plane parallel to the base.

The **altitude** of a frustum of a pyramid is the perpendicular distance between the planes of its bases.

The **slant height** of the frustum of a regular pyramid is the perpendicular distance, in a face, between the bases of that face.

A **truncated pyramid** is the part of a pyramid included between the base and a plane cutting all the lateral edges.

PRELIMINARY THEOREMS

598. THEOREM. The lateral edges of a regular pyramid are all equal. (504, II.)

599. THEOREM. The lateral faces of a regular pyramid are congruent isosceles triangles. (598, 78.)

600. THEOREM. The lateral edges of the frustum of a regular pyramid are all equal. (Ax. 2.)

601. THEOREM. The lateral faces of the frustum of a regular pyramid are congruent isosceles trapezoids. (484.)

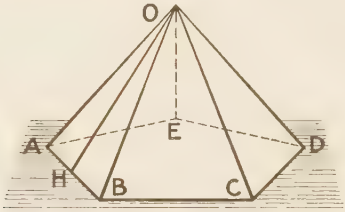
602. THEOREM. The lateral faces of the frustum of any pyramid are trapezoids. (?.)

603. THEOREM. The slant height of a regular pyramid is the same length in all the lateral faces.

THEOREMS AND DEMONSTRATIONS

Proposition XIV. Theorem

604. The lateral area of a regular pyramid is equal to half the product of the perimeter of the base by the slant height.



Given: Regular pyramid $O\text{–}ABCDE$; lateral area = to L; perimeter of base = to P; slant height OH = to s.

To Prove: $L = \frac{1}{2} P \cdot s$.

Proof:

Area $\triangle AOB = \frac{1}{2} AB \cdot s$
Area $\triangle BOC = \frac{1}{2} BC \cdot s$ (364).
etc. etc.

Adding, Lateral area $= \frac{1}{2} AB \cdot s + \frac{1}{2} BC \cdot s +$ etc. (Ax. 2).

That is, $L = \frac{1}{2}(AB + BC +$ etc.$) \cdot s$, or,

Lateral area, $L = \frac{1}{2} P \cdot s$ (Ax. 6). Q.E.D.

Ex. 1. Prove that the bases of any frustum of a pyramid are mutually equiangular.

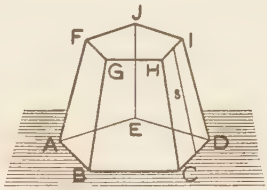
Ex. 2. The foot of the altitude of a regular pyramid drawn from the vertex, coincides with the center of the circles inscribed in, and circumscribed about, the base.

Ex. 3. The sum of the medians of the lateral faces of the frustum of a pyramid is equal to half the sum of the perimeters of the bases.

Ex. 4. To what rectangle is the lateral area of a regular pyramid equal? the total area?

Proposition XV. Theorem

605. The lateral area of the frustum of a regular pyramid is equal to half the sum of the perimeters of the bases multiplied by the slant height.



Given: (?).

To Prove: $L = \frac{1}{2}(P + p) \cdot s$.

Proof: Area trapezoid $CI = \frac{1}{2}(CD + HI) \cdot s$ (?).

Area trapezoid $BH = \frac{1}{2}(BC + GH) \cdot s$ (?).

Area trapezoid $AG = \frac{1}{2}(AB + FG) \cdot s$ (?) etc.

Adding, **Lateral area,** $L = \frac{1}{2}(P + p) \cdot s$ Explain.

Q.E.D.

Ex. 1. Using L for lateral area and B for area of base express the formula for the total area, T, of a regular pyramid.

Ex. 2. The slant height of a regular pyramid whose base is a square, of which each side is 8 ft., is 15 ft. Find the lateral area; the total area.

Ex. 3. A regular pyramid stands on a hexagonal base 16 in. on a side, and the slant height is 2 ft. Find the lateral and total areas.

Ex. 4. The slant height of the frustum of a regular pyramid is 12 in., and the bases are squares, 10 in. and 6 in. on a side, respectively. Find the lateral area and the total area.

Ex. 5. How many square feet of tin will be required to line a vat in the form of the frustum of a regular pyramid, having inside measurements as follows: the slant height is 14 ft.; the bases are regular hexagons whose sides are 7 ft. and 6 ft. respectively.

Ex. 6. The lateral edge of a pyramidal church spire is 61 ft. Each side of its octagonal base is 22 ft. What will be the cost of painting the spire at $2\frac{1}{2}¢$ a square foot?

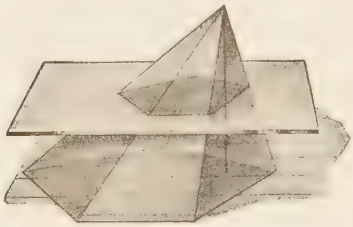
Proposition XVI. Theorem

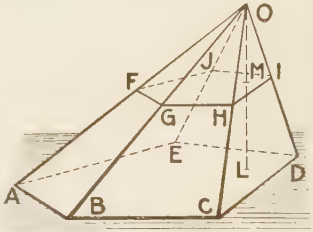
606. If a pyramid is cut by a plane parallel to the base:

I. The lateral edges and altitude are divided proportionally.

II. The section is a polygon similar to the base.

III. The area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.





Given: Pyr. O–$ABCDE$; plane $FI \parallel$ to the base; altitude $= OL$.

To Prove:

I. $\dfrac{OF}{OA} = \dfrac{OG}{OB} = \dfrac{OH}{OC} = \cdots = \dfrac{OM}{OL}$.

II. Section FI is similar to the base AD.

III. $\dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\overline{OM}^2}{\overline{OL}^2}$.

Proof: I. Imagine a plane through $O \parallel$ to plane AD.

This plane is $\perp$ to OL (496).

And $\parallel$ to plane FI (489).

$\therefore \dfrac{OF}{OA} = \dfrac{OG}{OB} = \dfrac{OH}{OC} = \cdots = \dfrac{OM}{OL}$ (500). Q.E.D.

II. FG is $\parallel$ to AB, GH is $\parallel$ to BC, etc. (484).

$\therefore \angle FGH = \angle ABC$; $\angle GHI = \angle BCD$; etc. (499).

That is, the polygons are mutually equiangular.

$\triangle OFG$ is similar to $\triangle OAB$; $\triangle OGH$, to $\triangle OBC$; etc. (305).

$$\therefore \frac{FG}{AB} = \left(\frac{OG}{OB}\right) = \frac{GH}{BC} = \left(\frac{OH}{OC}\right) = \frac{HI}{CD} = \text{etc.} \qquad (313).$$

$\therefore$ section FI is similar to base AD (301). Q.E.D.

III. Section FI is similar to base AD (II, above).

$$\frac{\text{section } FI}{\text{base } AD} = \frac{\overline{FG}^2}{\overline{AB}^2}. \qquad (376).$$

Now $\triangle OFG$ is similar to $\triangle OAB$ (305).

$$\therefore \frac{FG}{AB} = \frac{OF}{OA} \qquad (313).$$

But $$\frac{OF}{OA} = \frac{OM}{OL} \qquad \text{(I, above)}.$$

$$\therefore \frac{FG}{AB} = \frac{OM}{OL} \qquad (\text{Ax. } 1).$$

And $$\frac{\overline{FG}^2}{\overline{AB}^2} = \frac{\overline{OM}^2}{\overline{OL}^2} \qquad (287).$$

Hence $$\frac{\text{section } FI}{\text{base } AD} = \frac{\overline{OM}^2}{\overline{OL}^2} \qquad (\text{Ax. } 1). \quad \text{Q.E.D.}$$

Ex. 1. The bases of the frustum of a regular pyramid are equilateral triangles whose sides are 12 in. and 20 in., respectively. The slant height is 40 in. Find the lateral area; the total area.

Ex. 2. The bases of the frustum of a regular pyramid are regular hexagons whose sides are 8 in. and 18 in., respectively. The slant height is 25 in. Find the lateral area and total area.

Ex. 3. A pyramid whose altitude is 10 in. and whose base contains 80 sq. in. is cut by a plane bisecting the altitude and parallel to the base. Find the area of the section of the pyramid made by this plane.

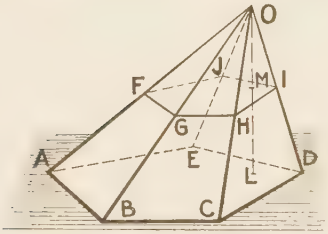
Ex. 4. Cutting a pyramid whose altitude is 16 ft. is a plane parallel to the base and 6 ft. from it. The area of the base is 192 sq. ft. What is the area of the section?

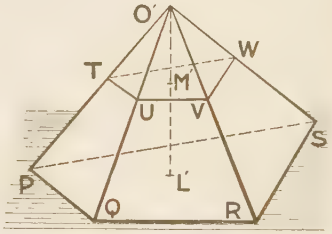
Ex. 5. If a plane parallel to the base of a pyramid bisects the altitude, how does the area of the section compare with the area of the base?

Ex. 6. Two planes, parallel to the base of a pyramid, trisect the altitude. How do the areas of the sections compare with the area of the base?

Proposition XVII. Theorem

607. **If two pyramids have equal altitudes and equal bases, sections made by planes parallel to the bases and at equal distances from the vertices are equal.**





Given: Pyramids O–$ABCDE$ and O'–$PQRS$; alt. $OL = O'L'$; base AD = base PR; planes of the sections FI and TV ∥ to the bases; $OM = O'M'$.

To Prove: Section FI = section TV.

Proof: $\dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\overline{OM}^2}{\overline{OL}^2}$ and $\dfrac{\text{section } TV}{\text{base } PR} = \dfrac{\overline{O'M'}^2}{\overline{O'L'}^2}$ (606, III).

But $OM = O'M'$ and $OL = O'L'$ (Hyp.).

$\therefore \dfrac{\text{section } FI}{\text{base } AD} = \dfrac{\text{section } TV}{\text{base } PR}$ (Ax. 1).

Now base AD = base PR (Hyp.).

Multiplying, section FI = section TV (Ax. 3).

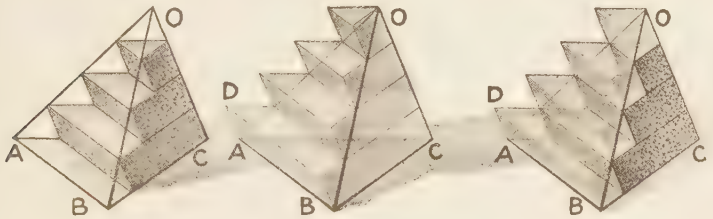
Q.E.D.

608. If a plane is passed parallel to the base of a pyramid, intersecting all the lateral edges, and upon this section, as a base, a prism is constructed wholly inside the pyramid, but having one lateral edge in a lateral edge of the pyramid, this prism is called an **inscribed prism**.

If upon this section as a base a prism is constructed partly outside the pyramid, having one lateral edge in one of the lateral edges of the pyramid, this prism is called a **circumscribed prism**.

Proposition XVIII. Theorem

609. The volume of a triangular pyramid is the limit of the sum of the volumes of a series of inscribed or circumscribed prisms, having equal altitudes, if the number of prisms is indefinitely increased.



Given: Triangular pyramid O–ABC, having a series of prisms inscribed in it, and another series circumscribed about it, all the prisms having equal altitudes.

To Prove: O–ABC is the limit of the sum of each series of prisms as their number is indefinitely increased.

Proof: Denote the volume of the pyramid by V, the sum of the volumes of the series of inscribed prisms by S_i, and the sum of the volumes of the series of circumscribed prisms by S_c.

The uppermost circumscribed prism = the uppermost inscribed prism (591).

The second pair of prisms also are equal (?).

And so on, until the last circumscribed prism, D–ABC, remains, for which there is no equivalent inscribed prism.

Hence it is evident that $S_c - S_i = D$–ABC, the lowest circumscribed prism.

Now by indefinitely increasing the number of the prisms, the altitude of D–ABC becomes indefinitely small, and hence the volume of D–ABC approaches zero as a limit.

The altitude can never actually equal zero, nor can the volume equal zero. Hence $S_c - S_i$ can be made less than any mentionable quantity, but cannot equal zero.

Now $S_c = S_c$	$V < S_c$ (Ax. 5).
and $\underline{V > S_i}$ (Ax. 5).	and $\underline{S_i = S_i}$.
$\therefore S_c - V < S_c - S_i$ (Ax. 9).	$\therefore V - S_i < S_c - S_i$ (Ax. 7).

That is, $S_c - V$ and $V - S_i$ are each less than $S_c - S_i$, which itself approaches zero.

Hence $S_c - V$ approaches zero and $V - S_i$ approaches zero.

$\therefore S_c$ approaches V as a limit, and S_i approaches V as a limit (227). (See note on p. xviii.)

Q.E.D.

Ex. 1. If a plane is passed parallel to the base of a pyramid, cutting the lateral edges, the section is to the base as the square of the lateral edge of the pyramid cut away by this plane is to the square of the lateral edge of the original pyramid. [See proof of 606, III.]

Ex. 2. If two pyramids have equal altitudes and are cut by planes parallel to the bases and at equal distances from the vertices, the sections formed will be to each other as the bases of the pyramids.

Ex. 3. In the figure of 609 prove that the planes of the faces of the prisms that are opposite OC are parallel to OC and AB.

Ex. 4. State the theorems leading up to the theorem of 575.

Ex. 5. State the theorems leading up to the theorem of 584.

Ex. 6. State the theorems leading up to the theorem of 590.

Ex. 7. The base of a pyramid is 180 sq. in. and its altitude is 15 in. What is the area of the section made by a plane parallel to the base, and 5 in. from the vertex?

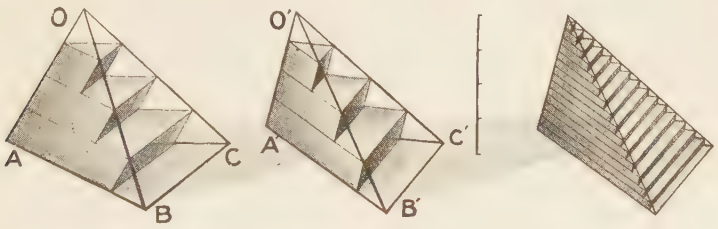
Ex. 8. The base of a pyramid is 200 sq. in., and its altitude is 12 in. At what distance from the vertex must a plane be passed so that the section shall contain half the area of the base?

Ex. 9. The base of a pyramid is B sq. in., and the altitude is H in. How far from the vertex must a plane parallel to the base be passed so that the area of the section shall contain $\frac{1}{3}$ B sq. in.? $\frac{1}{4}$ B sq. in.? Find the area of the section if the plane is passed $\frac{1}{3}$ H in. from the vertex; if it is passed $\frac{1}{4}$ H in. from the vertex.

Ex. 10. A pyramid having an altitude of 2 ft. and a base which is an equilateral triangle of 8 in. on a side, is cut by a plane parallel to the base and 18 in. from it. Find the area of the section.

Proposition XIX. Theorem

610. Two triangular pyramids having equal altitudes and equal bases are equal.



Given: Triangular pyramids O–ABC and O'–$A'B'C'$ having equal altitudes and base ABC = base $A'B'C'$.

To Prove: O–$ABC = O'$–$A'B'C'$.

Proof: Divide the altitude of each pyramid into any number of equal parts.

Through these points of division pass planes ∥ to the bases, forming triangular sections.

Upon these sections as bases construct inscribed prisms.

Denote the volumes of the pyramids by V and V', and the sums of the volumes of these series of prisms by S and S'.

The corresponding sections are equal (607).

$\therefore$ the corresponding prisms are equal (591).

Hence $S = S'$ (Ax. 2).

By indefinitely increasing the number of equal parts into which the altitudes are divided, the number of prisms becomes indefinitely great. $\therefore$ S approaches V as a limit (609).

And S' approaches V' as a limit (?).

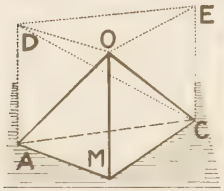
$\therefore$ $V = V'$ (229).

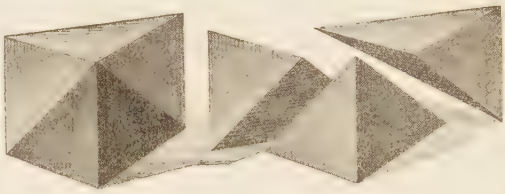
That is, O–$ABC = O'$–$A'B'C'$. Q.E.D.

Note. As in plane geometry, $\triangle ABC$ is the same as $\triangle BAC$, so in solid geometry the pyramid O–ABC is the same as the pyramid A–BCO or B–ACO or C–ABO.

Proposition XX. Theorem

611. The volume of a triangular pyramid is equal to one third the product of its base by its altitude.





Given: Triangular pyramid $O\text{–}AMC$, whose base $= B$ and altitude $= H$.

To Prove: Volume $O\text{–}AMC = \frac{1}{3} B \cdot H$.

Proof: Construct a prism $AMC\text{–}DOE$, having AMC as its base, and OM as one of its lateral edges. Pass a plane through DO and OC, cutting the face AE in line CD. The prism is now divided into three triangular pyramids.

In pyramids $O\text{–}AMC$ and $C\text{–}ODE$, the altitudes are $=$ (509).

The bases AMC and ODE are $\cong$ (553).

$\therefore$ pyramid $O\text{–}AMC =$ pyramid $C\text{–}ODE$ (610).

In pyramids $C\text{–}AMO$ and $C\text{–}AOD$, the altitudes are the same line, a $\perp$ from C to plane DM (491).

The bases AMO and AOD are $\cong$ (126).

$\therefore$ pyramid $C\text{–}AMO =$ pyramid $C\text{–}AOD$ (610).

Hence $O\text{–}AMC = C\text{–}ODE = C\text{–}AOD$ (Ax. 1).

That is, $O\text{–}AMC = \frac{1}{3}$ the prism.

But the volume of the prism $= B \cdot H$ (?).

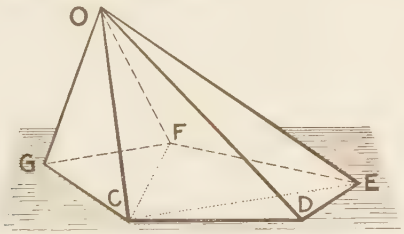
$\therefore$ volume of pyramid $O\text{–}AMC = \frac{1}{3} B \cdot H$ (Ax. 6). Q.E.D.

Ex. 1. In the figure of 611 prove pyramid $O\text{–}ACD = O\text{–}CDE$.

Ex. 2. The area of the base of a triangular pyramid is 30 sq. in., and its altitude is 20 in. Find the volume. Find the volume of the prism having the same base and altitude.

Proposition XXI. Theorem

612. The volume of any pyramid is equal to one third the product of its base by its altitude.



Given: Pyramid *O–CDEFG*, whose base $= B$ and altitude $= H$.

To Prove: Volume of *O–CDEFG* $= \frac{1}{3} B \cdot H$.

Proof: Through any lateral edge, *OC*, and lateral edges not adjoining *OC*, pass planes dividing the pyramid into triangular pyramids.

Vol. of *O–CFG* $= \frac{1}{3} CFG \cdot H$
Vol. of *O–CDE* $= \frac{1}{3} CDE \cdot H$ (611).
Vol. of *O–CEF* $= \frac{1}{3} CEF \cdot H$ Adding,
Vol. of *O–CDEFG* $= \frac{1}{3} B \cdot H$ (Ax. 2, 4). Q.E.D.

613. Corollary. Any two pyramids having equal altitudes and equal bases are equal. (Ax. 1.)

614. Corollary. Two pyramids having equal altitudes are to each other as their bases. (Prove.)

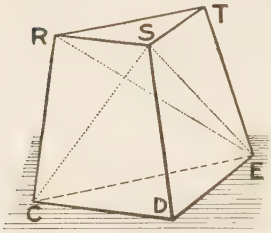
615. Corollary. Two pyramids having equal bases are to each other as their altitudes. (Prove.)

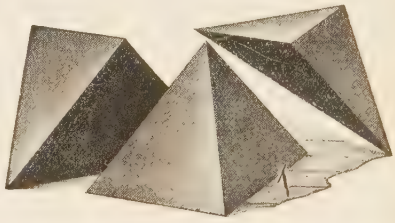
616. Corollary. Any two pyramids are to each other as the products of their bases by their altitudes. (Prove.)

Ex. The altitude of a pyramid is 15 ft., and each side of its square base is 8 ft. Find the volume. What is the volume of a prism having the same base and altitude?

Proposition XXII. Theorem

617. The volume of the frustum of a triangular pyramid is equal to one third the altitude multiplied by the sum of the lower base, the upper base and a mean proportional between the bases of the frustum.





Given: The frustum RD of a triangular pyramid whose lower base $= B$; upper base $= b$; altitude $= H$.

To Prove: Volume of $RD = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Proof: Pass a plane through edge CE and vertex S, and another through edge RS and vertex E, dividing the frustum into three triangular pyramids, $S\text{–}CDE$, $E\text{–}RST$, $E\text{–}CRS$.

I. $S\text{–}CDE = \frac{1}{3} H \cdot B$ (611).

II. $E\text{–}RST = \frac{1}{3} H \cdot b$ (?).

III. We shall now prove

$$E\text{–}CRS = \tfrac{1}{3} H \cdot \sqrt{B \cdot b}.$$

$$\frac{E\text{–}CSD}{E\text{–}CRS} = \frac{\triangle\, CSD}{\triangle\, CRS} \quad (614).$$

$$\frac{\triangle\, CSD}{\triangle\, CRS} = \frac{CD}{RS} \quad (368).$$

$$\therefore \frac{E\text{–}CSD}{E\text{–}CRS} = \frac{CD}{RS} \quad (\text{Ax. } 1).$$

Likewise $$\frac{S\text{–}CER}{S\text{–}ERT} = \frac{\triangle\, CER}{\triangle\, ERT} \quad (?).$$

And $$\frac{\triangle CER}{\triangle ERT} = \frac{CE}{RT} \qquad (?).$$

$$\therefore \frac{S\text{–}CER}{S\text{–}ERT} = \frac{CE}{RT} \qquad (?).$$

But $\triangle CDE$ and RST are similar (606, II).

$$\therefore \frac{CD}{RS} = \frac{CE}{RT} \qquad (?).$$

Hence $$\frac{E\text{–}CSD}{E\text{–}CRS} = \frac{S\text{–}CER \text{ or } E\text{–}CRS}{S\text{–}ERT} \qquad (Ax.\ 1).$$

That is, $$\frac{\frac{1}{2} H \cdot B}{E\text{–}CRS} = \frac{E\text{–}CRS}{\frac{1}{2} H \cdot b}$$

(Substituting from I and II).

$$\therefore E\text{–}CRS = \sqrt{\tfrac{1}{3} H \cdot B \cdot \tfrac{1}{3} H \cdot b} = \tfrac{1}{3} H \cdot \sqrt{B \cdot b} \qquad (289).$$

$$\therefore \text{Volume of the frustum} = \tfrac{1}{3} H[B + b + \sqrt{B \cdot b}] \qquad (Ax.\ 2).$$

Q.E.D.

Note. The theorem of 617 is sometimes stated thus:

The frustum of a triangular pyramid is equal to the sum of three pyramids whose altitudes are the same as the altitude of the frustum and whose bases are the lower base, the upper base, and a mean proportional between the bases of the frustum.

Ex. 1. Find the volume of the pyramid whose altitude is 18 in. and whose base is 10 in. square.

Ex. 2. Find the volume of the frustum of a triangular pyramid whose altitude is 20 in. and the areas of whose bases are 18 sq. in. and 32 sq. in.

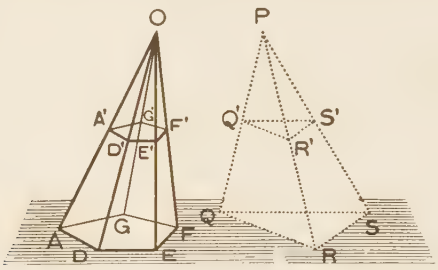
Ex. 3. If, in the formula for the volume of the frustum of a triangular pyramid, $b = B$, show that the formula becomes the correct formula for the volume of a prism. And, again, if $b = 0$, show that the formula of 617 becomes the correct formula for the volume of a pyramid.

Ex. 4. What is the locus of the vertices of all pyramids upon the same base and having the same volume?

Historical Note. The Greeks, as early as the fourth century B.C., knew that the pyramid is the third part of the prism having the same base and altitude. Eudoxus, an Athenian mathematician, is given credit for the discovery and proof of this truth.

Proposition XXIII. Theorem

618. The volume of the frustum of any pyramid is equal to one third the altitude multiplied by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.



Given: Pyr. O–$ADEFG$; frustum $A'F$, whose lower base $= B$, upper base $= b$, altitude $= H$.

To Prove: Volume of frustum $= \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Proof: Construct a $\triangle QRS =$ to polygon AF (395).

Upon $\triangle QRS$ as a base, construct a pyramid whose altitude $=$ the altitude of O–$ADEFG$.

Pass a plane $Q'R'S' \perp$ to QRS and at a distance from $QRS =$ to H.

Vol. of $Q'R = \frac{1}{3} H [\triangle QRS + \triangle Q'R'S' + \sqrt{\triangle QRS \cdot \triangle Q'R'S'}]$ (617).

The alt. of P–$Q'R'S' =$ alt. of O–$A'D'E'F'G'$ (Ax. 2).

Also $QRS = B$ (Const.); and $Q'R'S' = b$ (607).

$\therefore$ vol. of O–$ADEFG =$ vol. of P–QRS (613).

And vol. of O–$A'D'E'F'G' =$ vol. of P–$Q'R'S'$ (?).

Subtracting, vol. of frustum $A'F =$ vol. of frustum $Q'R$ (Ax. 2).

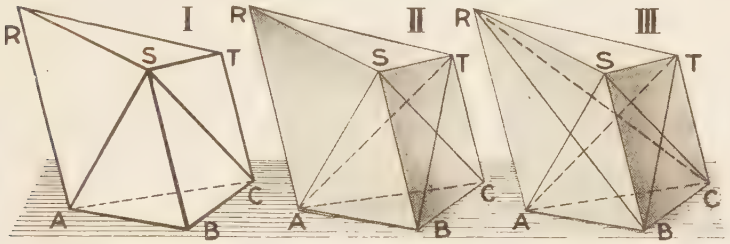
Vol. of frustum $A'F = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$ (Ax. 6).

Q.E.D.

Ex. The bases of the frustum of a pyramid are regular hexagons whose sides are 10 in. and 6 in., respectively. The altitude of the frustum is 2 ft. Find its volume.

Proposition XXIV. Theorem

619. **A truncated triangular prism is equal to three triangular pyramids whose bases are the base of the prism and whose vertices are the three vertices of the face opposite the base (the inclined section).**



Given: The truncated triangular prism $ABC\text{–}RST$, whose base is ABC and whose opposite vertices are R, S, T. Let it be divided by the planes ACS, ABT, BCR.

To Prove: $ABC\text{–}RST = R\text{–}ABC + S\text{–}ABC + T\text{–}ABC$ (III).

Proof: In Fig. I, $S\text{–}ABC$ is obviously one of these pyramids.

In Fig. II, $A\text{–}CST = A\text{–}BCT$ (613).

That is, $A\text{–}CST = T\text{–}ABC$.

In Fig. III, $T\text{–}ARS = T\text{–}ABR$ (613).

$T\text{–}ABR = C\text{–}ABR$ (613).

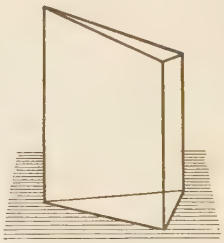
$\therefore T\text{–}ARS = R\text{–}ABC$ (Ax. 1).

Now $ABC\text{–}RST = T\text{–}ARS + S\text{–}ABC + A\text{–}CST$ (Ax. 4).

Hence $ABC\text{–}RST = R\text{–}ABC + S\text{–}ABC + T\text{–}ABC$ (Ax. 6).

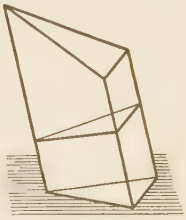
Q.E.D.

Ex. There are approximately $1\frac{1}{4}$ cu. ft. in a bushel. Find the capacity, in bushels, of a grain elevator, 30 ft. high, in the shape of the frustum of a square pyramid, and having bases 24 ft. square and 16 ft. square.



620. Corollary. **The volume of a truncated triangular prism is equal to the product of the base by one third the sum of the three altitudes drawn to the base from the three vertices opposite the base.**

621. Corollary. **The volume of a truncated right triangular prism is equal to the product of the base by one third the sum of its lateral edges.**



622. Corollary. **The volume of any truncated triangular prism is equal to the product of its right section by one third the sum of its lateral edges.**

Proof: The right section divides the solid into two truncated right prisms.

Hence the volume = the right section $\times \frac{1}{3}$ the sum of the lateral edges. (621.) Q.E.D.

Ex. 1. A pyramid whose volume is V and whose altitude is H, is bisected by a plane parallel to the base. Find the distance of this plane from the vertex.

Ex. 2. The altitude of a square pyramid each side of whose base is 6 ft., is 10 ft. Parallel to the base and 2 ft. from the vertex a plane is passed. Find the area of the section. Find the volumes of the two pyramids concerned, and hence find the volume of the frustum.

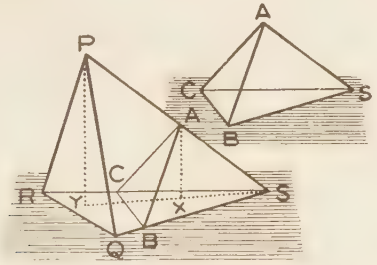
Ex. 3. The base of a pyramid is a rhombus whose diagonals are 7 m. and 10 m. Find the volume if the altitude is 15 m.

Ex. 4. The areas of the bases of the frustum of a pyramid are 3 sq. ft. and 27 sq. ft. The volume is 104 cu. ft. Find the altitude.

Ex. 5. State what distances must be known in order to find the volume of a truncated right triangular prism, and of a truncated oblique triangular prism. Now explain how to use these lengths to find the volume.

Proposition XXV. Theorem

623. Two triangular pyramids (tetrahedrons) having a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges including the equal trihedral angles.



Given: Triangular pyramids $S\text{–}ABC$, $S\text{–}PQR$; having the trih. $\measuredangle$ at S equal; and their volumes V and V'.

To Prove: $\dfrac{V}{V'} = \dfrac{SA \cdot SB \cdot SC}{SP \cdot SQ \cdot SR}$.

Proof: Place the pyramids so that the equal trihedral $\measuredangle$ coincide. Draw the altitudes AX and PY and the projection SXY, of the edge PS, in plane SQR.

$\triangle SAX$ is similar to $\triangle SPY$ (304).

Now $\dfrac{V}{V'} = \dfrac{\triangle SBC \cdot AX}{\triangle SQR \cdot PY} = \dfrac{\triangle SBC}{\triangle SQR} \cdot \dfrac{AX}{PY}$ (616).

But $\dfrac{\triangle SBC}{\triangle SQR} = \dfrac{SB \cdot SC}{SQ \cdot SR}$ (374) and $\dfrac{AX}{PY} = \dfrac{SA}{SP}$ (313).

Hence, by substituting in the first equation above,

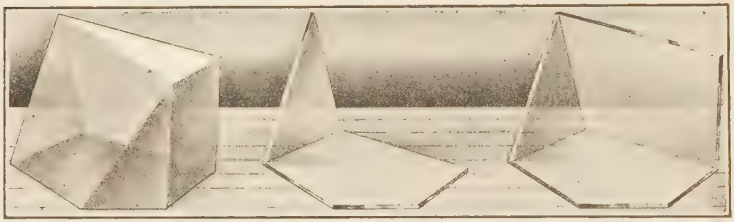
$$\frac{V}{V'} = \frac{SB \cdot SC}{SQ \cdot SR} \cdot \frac{SA}{SP} = \frac{SA \cdot SB \cdot SC}{SP \cdot SQ \cdot SR} \text{ (Ax. 6). Q.E.D.}$$

Ex. 1. To what is the ratio of $\triangle SAB$ to $\triangle SPQ$ equal?

Ex. 2. Reduce the formula of 623 if $SA = SB = SC$ and $SP = SQ = SR$.

Proposition XXVI. Theorem

624. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.



Given: A polyhedron; E = number of edges; F = number of faces; V = number of vertices.

To Prove: $E + 2 = V + F$.

Proof: Suppose the surface of the polyhedron is put together, face by face.

For one face, $E = V$ (145). (Begin with the base.)

By attaching an adjoining face, the number of edges is one greater than the number of vertices.

That is, for two faces, $E = V + 1$.
For three faces, $E = V + 2$.
For four faces, $E = V + 3$.
For five faces, $E = V + 4$.
. . . .
For n faces, $E = V + (n - 1)$.
For $F - 1$ faces, $E = V + (F - 2)$.

By attaching the last face, neither the number of edges nor the number of vertices is increased.

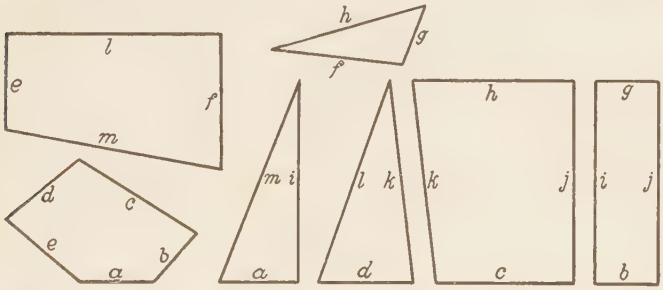
That is, for F faces, $E = V + F - 2$.

$\therefore$ for the complete solid, $E + 2 = V + F$ (Ax. 2). Q.E.D.

625. Corollary. In any polyhedron the difference between the number of edges and the number of faces is two less than the number of vertices, that is, $E - F = V - 2$.

Proposition XXVII. Theorem

626. The sum of all the face angles of any polyhedron is equal to 4 right angles multiplied by two less than the number of vertices, that is, $S\measuredangle = (V-2)\,4$ rt. $\measuredangle = (V-2)\,360°$.



Given: A polyhedron; E = number of edges; F = number of faces; V = number of vertices.

To Prove: Sum of all the face $\measuredangle = (V-2)\ 4$ rt. $\measuredangle$.
Or, $S\measuredangle = (V-2)\ 360°$.

Proof: If the faces are considered as separate polygons, it is obvious that each edge is a side of two polygons, that is, the number of sides of the several faces $= 2E$.

$\therefore$ the number of vertices of all the polygons $= 2E$ (145).
Suppose an exterior $\angle$ formed at each of these $2E$ vertices.
Then the sum of the exterior $\measuredangle$ of each face $= 4$ rt. $\measuredangle$ (?).
Sum of int. and ext. $\measuredangle$ at each vertex $= 2$ rt. $\measuredangle$ (?).

$\therefore$ the int. $\measuredangle$ + ext. $\measuredangle$ at all the $2E$ vertices $= 4E$ rt. $\measuredangle$
But the ext. $\measuredangle$ of all the F faces $= 4F$ rt. $\measuredangle$ (?).

$\therefore$ the int. $\measuredangle$ of these polygons $= 4E$ rt. $\angle - 4F$ rt. $\measuredangle$ (?).
$= (E-F)\cdot 4$ rt. $\measuredangle$.
But $E - F = V - 2$ (625).
$\therefore$ $S\measuredangle = (V-2)\ 4$ rt. $\angle = (V-2)\ 360°$ (Ax. 6). Q.E.D.

627. Prismatoid. A prismatoid is a polyhedron two of whose faces, called **bases**, are polygons in parallel planes, and

whose other faces are triangles, trapezoids, or parallelograms.

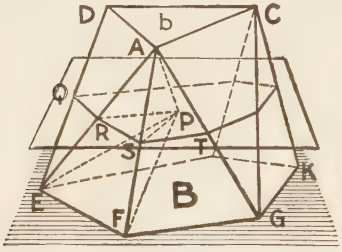
The **altitude** of a prismatoid is the distance between the planes of the bases.

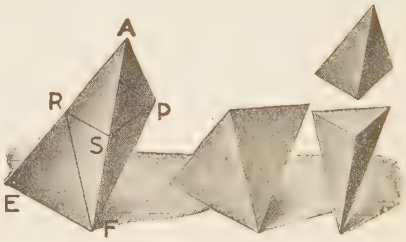
The **mid-section** of a prismatoid is the section made by a plane parallel to the bases and bisecting the altitude.

A **prismoid** is a prismatoid whose lateral faces are either trapezoids or parallelograms.

Proposition XXVIII. Theorem

628. The volume of a prismatoid is equal to the product of one sixth of its altitude by the combined sum of the two bases and four times the mid-section.





Given: The prismatoid *ACD–EFGKT*, with bases b and B, mid-section M, altitude H, and volume V.

To Prove: $V = \frac{1}{6} H[b + B + 4M]$.

Proof: Through any point, P, in the mid-section, pass planes, each containing an edge of the solid.

These planes will divide the prismatoid into pyramids:

I. The pyramid P–ACD, whose vertex is P, base b, altitude $\frac{1}{2} H$.

Of this pyramid, $v_1 = \frac{1}{6} b \cdot H$ (612).

II. The pyramid P–$EFGKT$, whose vertex is P, base B, altitude $\frac{1}{2} H$.

Of this pyramid, $v_2 = \frac{1}{6} BH$.

III. Several pyramids, like $P\text{–}AEF$, whose combined volume $= \frac{4}{6}\, M \cdot H$, as we shall now prove. $AE = 2\, AR$ (?).

$\therefore\ \triangle\, AEF = 4 \triangle\, ARS$ (376).

$\therefore$ pyramid $P\text{–}AEF = 4 \cdot$ pyramid $P\text{–}ARS$ (614).

But pyramid $P\text{–}ARS = \frac{1}{3} \triangle\, PRS \cdot \frac{1}{2}\, H = \frac{1}{6} \triangle\, PRS \cdot H$ (611).

That is, pyramid $P\text{–}AEF = \frac{4}{6} \triangle\, PRS \cdot H$.

Hence, for the sum of all such pyramids in the prismatoid,

$v_3 = \frac{4}{6}\, M \cdot H$ (Ax. 4).

By addition, $V = \frac{1}{6}\, bH + \frac{1}{6}\, BH + \frac{4}{6}\, MH$ (Ax. 2).

Or, $\boldsymbol{V = \frac{1}{6} H[b + B + 4\, M].}$ Q.E.D.

Ex. 1. A prismatoid has an upper base 5 sq. in., a lower base 11 sq. in., a mid-section 8 sq. in., and an altitude 9 in. Find the volume.

Ex. 2. How does the mid-section of a prism compare with the base? the mid-section of a pyramid? of a cube?

Ex. 3. Will the formula for the volume of a prismatoid give the volume of a cube? Will it give the formula for the volume of a prism?

Ex. 4. Reduce the prismatoid formula to a formula for the volume of a parallelepiped.

Ex. 5. Derive the formula for the volume of a pyramid from the prismatoid formula.

Ex. 6. Is a prism a prismatoid? is a pyramid? is a truncated prism? is the frustum of a pyramid? is a parallelepiped?

Historical Note. The prismatoid formula was discovered by a German, E. F. August, in the latter part of the nineteenth century. Its importance in the mensuration of polyhedrons was recognized at once by mathematicians throughout the world.

REGULAR AND SIMILAR POLYHEDRONS

629. A **regular polyhedron** is a polyhedron whose faces are congruent regular polygons and whose polyhedral angles are all equal.

Similar polyhedrons are polyhedrons which have the same number of faces similar each to each and similarly placed, and which have their homologous polyhedral angles equal.

Proposition XXIX. Theorem

630. There can exist no more than five kinds of regular polyhedrons.

Proof: The faces must be equilateral △, squares, regular pentagons, or some other regular polygons. (629.)

There must be at least three faces at each vertex. (565.)

The sum of the face ∠ at each vertex is $< 360°$. (549.)

I. Each ∠ of an equilateral $\triangle = 60°$ (?). Hence we may form a polyhedral ∠ by placing 3, 4, or 5 equilateral △ at a vertex, but not 6 (?). That is, only three regular polyhedrons can be formed having equilateral triangles for faces.

II. Each ∠ of a square $= 90°$ (?). Hence we may form a polyhedral ∠ by placing 3 squares at a vertex; but not 4. That is, only one regular polyhedron can be formed having squares for faces.

III. Each ∠ of a regular pentagon $= 108°$ (155). Hence we may form a polyhedral ∠ by placing 3, but not 4 regular pentagons at a vertex. That is, only one regular polyhedron can be formed having regular pentagons for faces.

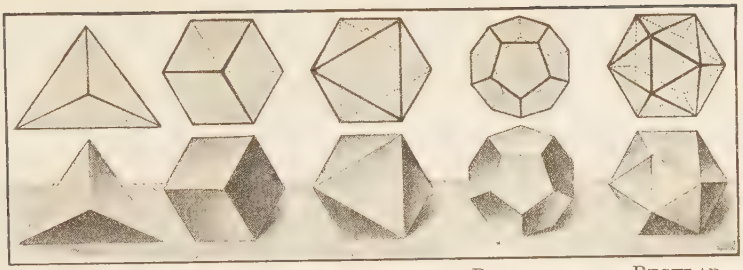
IV. Each ∠ of a regular hexagon $= 120°$ (?).

$\therefore$ no polyhedral ∠ can be formed by hexagons (?).

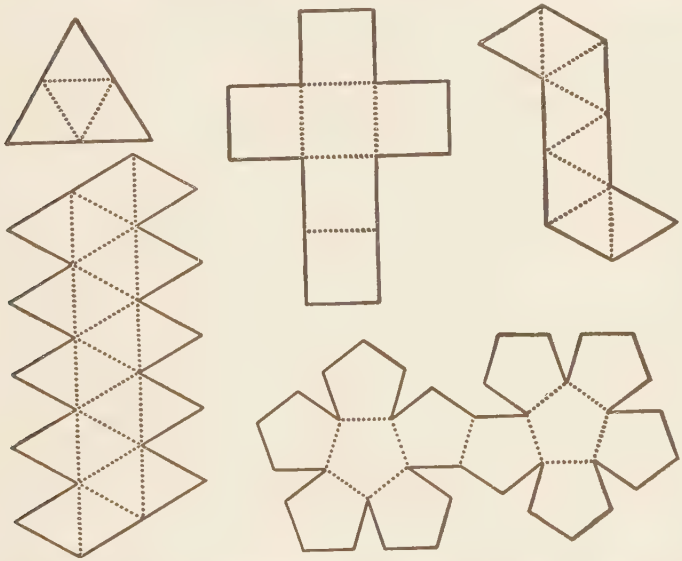
Consequently there can be no more than five kinds of regular polyhedrons, — three kinds bounded by triangles, one kind by squares, and one by pentagons. Q.E.D.

631. The **names** of the regular polyhedrons.

Names	Total Number of Faces	Number of Faces at Each Vertex	Kinds of Faces
Regular tetrahedron	4	3	Equilateral triangles
Regular hexahedron (cube)	6	3	Squares
Regular octahedron	8	4	Equilateral triangles
Regular dodecahedron	12	3	Regular pentagons
Regular icosahedron	20	5	Equilateral triangles



REGULAR TETRAHEDRON | CUBE | REGULAR OCTAHEDRON | REGULAR DODECAHEDRON | REGULAR ICOSAHEDRON



DIRECTIONS FOR CONSTRUCTION. — Mark on cardboard larger figures similar to the drawings. Cut the dotted lines half through and the solid lines entirely through. Fold along the dotted lines, closing the solids up and forming the figures. Paste strips of paper along the edges.

Historical Note. Pythagoras knew about the existence of all the regular polyhedrons except the dodecahedron. This was discovered in 470 B.C. by Hippasus, who having boasted of his discovery was drowned by the other Pythagoreans. The regular polyhedrons were supposed to have certain magical properties and their study was greatly emphasized.

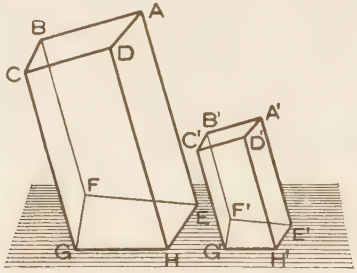
Proposition XXX. Theorem

632. In two similar polyhedrons:

I. Homologous edges are proportional.

II. Homologous faces are to each other as the squares of any two homologous edges.

III. Total surfaces are to each other as the squares of any two homologous edges.



Proof: I. Homologous faces are similar (629).

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DH}{D'H'} = \frac{AE}{A'E'} = \frac{BF}{B'F'} = \text{etc.} \quad (313).$$

II. $$\frac{\text{Face } DG}{\text{Face } D'G'} = \frac{\overline{DH}^2}{\overline{D'H'}^2} = \frac{\text{face } AH}{\text{face } A'H'^2} = \frac{\overline{AE}^2}{\overline{A'E'}^2} = \text{etc.} \quad (376).$$

III. $$\frac{\text{Face } DG}{\text{Face } D'G'} = \frac{\text{face } AH}{\text{face } A'H'} = \frac{\text{face } GE}{\text{face } G'E'} = \text{etc.} \quad (\text{Ax. } 1).$$

$$\frac{\text{Total surface of } AG}{\text{Total surface of } A'G'} = \frac{\text{face } DG}{\text{face } D'G'} = \frac{\overline{DH}^2}{\overline{D'H'}^2} = \frac{\overline{AE}^2}{\overline{A'E'}^2} = \text{etc.} \quad (291).$$

Or, $$T : T' = e^2 : e'^2.$$ Q.E.D.

633. Corollary. If a pyramid is cut by a plane parallel to the base, the pyramid cut away is similar to the original pyramid. (629.)

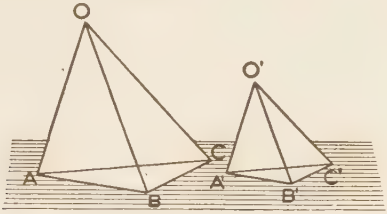
Ex. 1. Show that the theorem of 624 holds true in the case of a cube.

Ex. 2. Show that the theorem of 626 holds true in the case of a cube.

Ex. 3. Show that 624 and 626 are true of a regular octahedron.

Proposition XXXI. Theorem

634. Two similar tetrahedrons are to each other as the cubes of any two homologous edges.



Given: Similar tetrahedrons $O\text{–}ABC$ and $O'\text{–}A'B'C'$; whose volumes $= V$ and V'.

To Prove: $V : V' = \overline{AB}^3 : \overline{A'B'}^3 = \text{etc.}$

Proof: Trihedral $\angle A =$ trihedral $\angle A'$ (629).

$$\therefore \frac{V}{V'} = \frac{AB \cdot AC \cdot AO}{A'B' \cdot A'C' \cdot A'O'} \quad (623).$$

$$= \frac{AB}{A'B'} \cdot \frac{AC}{A'C'} \cdot \frac{AO}{A'O'}.$$

But $$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{AO}{A'O'} \quad (632, \text{I}).$$

$$\therefore \frac{V}{V'} = \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} \cdot \frac{AB}{A'B'} \quad (\text{Ax. } 6).$$

That is, $V : V' = \overline{AB}^3 : \overline{A'B'}^3 = \overline{AC}^3 : \overline{A'C'}^3 = \text{etc.}$ Q.E.D.

Or, $$V : V' = e^3 : e'^3.$$

Ex. 1. If, in the figure of 634, $AO = 4$ in. and $A'O' = 1$ in., what is the ratio of the total surfaces of the tetrahedrons? of their volumes?

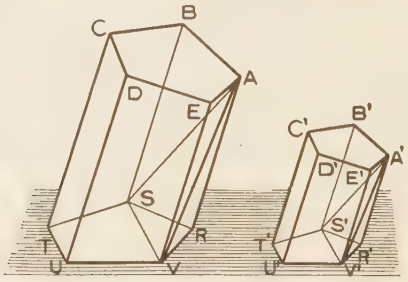
Ex. 2. Two homologous edges of two similar polyhedrons are 2 in. and 3 in. What is the ratio of their total surfaces?

Ex. 3. Two homologous edges of two similar tetrahedrons are 2 in. and 5 in. The total surface of the less is 28 sq. in., and its volume is 40 cu. in. Find the area of the total surface and the volume of the other.

Ex. 4. Show that the theorems of 624 and 626 are true in the cases of regular dodecahedrons and regular icosahedrons.

Proposition XXXII. Theorem

635. Two similar polyhedrons can be decomposed into the same number of tetrahedrons similar each to each and similarly placed.



Given: (?).

To Prove: (?).

Proof: Suppose diagonals drawn in every face of AT, except the faces containing vertex A, dividing the faces into $\triangle$. (The figure shows only SV.)

Suppose lines drawn from A to the several vertices of these $\triangle$. (The figure shows only AS, AV.)

Obviously, this process divides the solid (by planes) into tetrahedrons, each of which has a vertex at A.

Then construct homologous lines in solid $A'T'$. There will evidently be as many lines in $A'T'$ as in AT and as many tetrahedrons, and these will be similarly placed.

Now, in the tetrahedrons A–SVR and A'–$S'V'R'$,

$\triangle AVR$ is similar to $\triangle A'V'R'$;

$\triangle ASR$ is similar to $\triangle A'S'R'$;

$\triangle SVR$ is similar to $\triangle S'V'R'$ (318).

Also $$\frac{AV}{A'V'} = \frac{VR}{V'R'} = \frac{VS}{V'S'} = \frac{RS}{R'S'} = \frac{AS}{A'S'} \qquad (313).$$

$$\therefore \frac{AV}{A'V'} = \frac{VS}{V'S'} = \frac{AS}{A'S'} \qquad (Ax.\ 1).$$

Hence $\triangle ASV$ is similar to $\triangle A'S'V'$ (308).

Also the trihedral $\measuredangle R$ and R' are equal; S and S' are equal; V and V' are equal, etc. (546).

$\therefore$ the two tetrahedrons are similar (Def. 629).

Furthermore, after removing these tetrahedrons, the remaining polyhedrons are similar. (Def. 629.)

By the same process other pairs of tetrahedrons may be removed and proved similar, and the process may be continued until the polyhedrons are completely decomposed into tetrahedrons similar each to each and similarly placed. Q.E.D.

Proposition XXXIII. Theorem

636. The volumes of two similar polyhedrons are to each other as the cubes of any two homologous edges.

Given: Similar polyhedrons AT and $A'T'$; volumes V and V'; AR and $A'R'$, any two homologous edges. (See figure of 635.)

To Prove: (?).

Proof: These solids may be decomposed, etc. (635).

Denote the volumes of tetrahedrons of AT by w, x, y, z, etc.; of $A'T'$ by w', x', y', z', etc.

Then $\frac{w}{w'}=\frac{\overline{AR}^3}{\overline{A'R'}^3}$; $\frac{x}{x'}=\frac{\overline{AR}^3}{\overline{A'R'}^3}$; $\frac{y}{y'}=\frac{\overline{AR}^3}{\overline{A'R'}^3}$; etc. (634).

Hence $\frac{w}{w'}=\frac{x}{x'}=\frac{y}{y'}=\text{etc.}$ (Ax. 1).

Therefore $\frac{w+x+y+\text{etc.}}{w'+x'+y'+\text{etc.}}=\frac{w}{w'}$ (291).

That is, $\frac{V}{V'}=\frac{\overline{AR}^3}{\overline{A'R'}^3}$ (Ax. 6). Q.E.D.

637. Corollary. The volumes of two similar pyramids are to each other as the cubes of their altitudes. (Explain.)

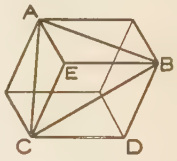
ORIGINAL EXERCISES

1. What plane through the vertex of a given tetrahedron will divide it into two equal parts? Prove.

2. The area of the base of any pyramid is less than the sum of the lateral faces.

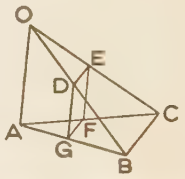
[Draw the altitudes of the lateral faces and the projections of the altitudes upon the base.]

3. Three of the edges of a parallelepiped that meet in a point are also the lateral edges of a pyramid. What part of the parallelepiped is this pyramid?



4. A plane is passed containing one vertex of a parallelepiped and a diagonal of a face not containing that vertex. What part of the volume of the parallelepiped is the pyramid thus cut off?

5. Any section of a tetrahedron made by a plane parallel to two opposite edges, is a parallelogram.



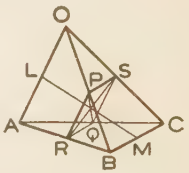
Given: Section $DEFG \parallel$ to OA and BC.

To Prove: $DEFG$ is a $\square$.

Proof: EF is $\parallel$ to OA, DG is $\parallel$ to OA (?). Also DE is $\parallel$ to BC and GF is $\parallel$ to BC (?), etc.

6. The three lines that join the midpoints of the opposite edges of a tetrahedron meet in a point and bisect one another.

Given: LM, PQ, RS, three lines, etc. **To Prove**: (?).



Proof: Join PS, SQ, QR, PR. PS is $\parallel$ to and $= \frac{1}{2} BC$; RQ is $\parallel$ to and $= \frac{1}{2} BC$. (Explain.)

$\therefore$ fig. $PSQR$ is a $\square$ (?).

Similarly, discuss LM and SR.

7. A pyramid having one of the faces of a cube for its base and the center of the cube for its vertex, contains one sixth of the volume of the cube.

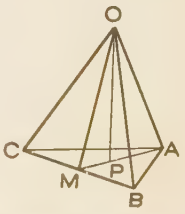
8. A plane containing an edge of a regular tetrahedron and the midpoint of the opposite edge:

(a) contains the medians of two faces;

(b) is perpendicular to the opposite edge;

(c) is perpendicular to these two faces;

(d) contains two altitudes of the tetrahedron.

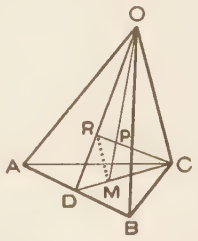


9. The altitude of a regular tetrahedron meets the base at the point of intersection of the medians of the base.

10. The altitude of a regular tetrahedron $= \frac{1}{3}\sqrt{6}$ times the edge.

11. The altitudes of a regular tetrahedron meet at a point.

12. The lines joining the vertices of *any* tetrahedron to the point of intersection of the medians of the opposite face meet in a point that divides each line into segments in the ratio 3 : 1.



Given: OM, CR, two such lines.

To Prove: The four such lines meet, etc.

Proof: OM and CR lie in the plane determined by OC and point D, the midpoint of AB.

$\therefore$ OM and CR intersect. Draw RM.

$DR = \frac{1}{2}RO$, $DM = \frac{1}{2}MC$ (?). $\therefore \frac{DR}{DM} = \frac{RO}{MC}$ (?).

$\therefore RM$ is $\parallel$ to OC (?).

$\therefore$ $\triangle DMR$ and DCO are similar (?); and $DR : DO = RM : OC$. (?)

Thus $RM = \frac{1}{3}OC$. (Explain.)

Also $\triangle PRM$ and OPC are similar (?).

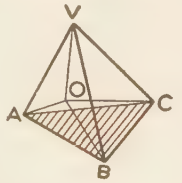
Hence $OP : PM = OC : RM = 3 : 1$.

And $CP : PR = OC : RM = 3 : 1$. (Explain.)

Q.E.D.

NOTE. This point P is called the **center of gravity** of the tetrahedron.

13. There can be no polyhedron having seven edges and only seven.



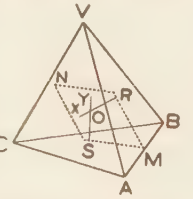
14. The planes bisecting the dihedral angles of any tetrahedron meet in a point that is equally distant from the faces.

15. The lines perpendicular to the faces of any tetrahedron, at the centers of the circles circumscribed about the faces, meet in a point that is equally distant from the vertices.

Proof: RX and SY are loci of points, etc. (511).

Plane MN, $\perp$ to AB at M, the midpoint of AB, is the locus of points, etc.

$\therefore$ RX and SY lie in MN and intersect at O, etc.

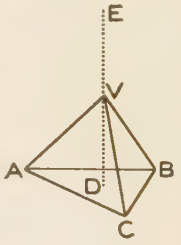


16. If a plane is passed through the midpoints of the three edges of a parallelepiped that meet at a vertex, what part of the whole solid is the pyramid thus cut off?

17. The plane bisecting a dihedral angle of a tetrahedron divides the opposite edge into two segments proportional to the areas of the faces that form the dihedral angle.

18. Two tetrahedrons are similar if a dihedral angle of one equals a dihedral angle of the other and the faces forming these dihedral angles are respectively similar.

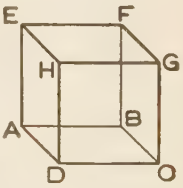
19. If from any point within a regular tetrahedron perpendiculars to the four faces are drawn, their sum is constant and equal to the altitude of the tetrahedron.



20. Construct a regular tetrahedron upon a given edge.

Construction: Upon AB, construct an equilateral $\triangle ABC$. Erect $ED \perp$ to plane of $\triangle ABC$, at D, the center of circumscribed $\odot$. Take V on ED such that $AV = BV = CV = AB$, etc.

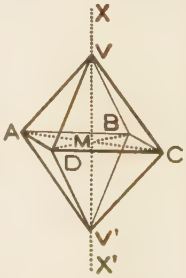
21. Construct a regular hexahedron upon a given edge.



Construction: Upon AB construct a square $ABCD$. At the vertices erect $\perp$s $= AB$ and join the extremities, etc.

22. Construct a regular octahedron upon a given edge.

Construction: Upon AB construct a square $ABCD$. At M, the center of the square, erect $XX' \perp$ to plane of $ABCD$. On XX' take $MV = MV' = MD$. Draw the edges from V and V'.



Statement: VV' is a regular octahedron.

Proof: The right $\triangle DMV$, DMC, DMV' are equal. (Explain.)

Thus the 12 edges are equal and the 8 faces are equal. (Explain.)

Figures $AVCV'$, $DVBV'$, $ABCD$ are equal squares. (Explain.)

Then, pyramids V–$ABCD$, D–$AVCV'$, etc., are equal and the 6 polyhedral angles are equal. (Explain.) $\therefore$ etc.

23. Pass a plane through a cube so that the section will be a regular hexagon.

24. Pass planes through three given lines in space, no two of which are parallel, which shall inclose a parallelepiped.

25. Find the lateral area and the total area of a regular pyramid whose slant height is 20 in. and whose base is a square, 1 ft. on a side.

26. Find the volume of a pyramid whose altitude is 18 in. and whose base is an equilateral triangle each side of which is 8 in.

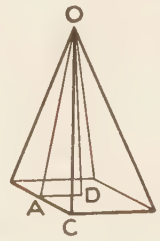
27. A regular hexagonal pyramid has an altitude of 9 ft. and each edge of the base is 6 ft. Find the volume.

28. The base of a pyramid is an isosceles triangle whose sides in inches are 14, 25, 25, and the altitude of the pyramid is 12 in. Find its volume.

29. The altitude of the frustum of a pyramid is 25 in., and the bases are squares whose sides are 4 in. and 10 in., respectively. Find the volume of the frustum.

30. The frustum of a regular pyramid has hexagons for bases whose sides are 5 in. and 9 in., respectively. The slant height of the frustum is 14 in. Find its lateral area. Find its total area.

31. The altitude of a regular pyramid is 15 in., and each side of its square base is 16 in. Find the slant height, the lateral edge, the total area, and the volume.



$\overline{OA}^2 = \overline{OD}^2 + \overline{DA}^2 = (15)^2 + (8)^2 = 289.$

$\therefore AO = 17.$ $OC^2 = OA^2 + AC^2 = 289 + 64 = 353.$

$\therefore OC = \sqrt{353} = 18.78+.$

32. The slant height of a regular pyramid is 39 ft., the altitude is 36 ft., and the base is a square. Find the lateral area and the volume.

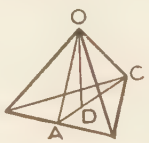
33. The lateral edge of a regular pyramid is 37 in. and each side of the hexagonal base is 12 in. Find the slant height, the lateral area, the total area, and the volume.

In rt. $\triangle ACD$, $CD = 12$, $AC = 6$, $\therefore AD = 6\sqrt{3}$.

In rt. $\triangle ACO$, $CO = 37$, $AC = 6$, $\therefore AO = \sqrt{1333}$.

In rt. $\triangle CDO$, $CO = 37$, $CD = 12$, $\therefore OD = 35$, etc.

34. Find the total area and volume of a regular tetrahedron whose edge is 6 in.

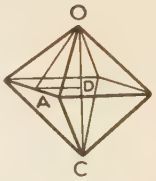


The four faces are equal equilateral $\triangle$. $\therefore AO = AC = 3\sqrt{3}$ in.; $\therefore AD = \sqrt{3}$ in. and $CD = 2\sqrt{3}$ in.

Hence $OD = 2\sqrt{6}$ in. Area of any face $= 9\sqrt{3}$ sq. in., etc.

35. Find the total area and the volume of a regular tetrahedron whose edge is 10 in.

36. Find the total area and the volume of a regular hexahedron whose edge is 8 in.



37. Find the total area and the volume of a regular octahedron whose edge is 16 in.

The 8 faces are equal equilateral $\triangle$. $AO = 8\sqrt{3}$. In $\triangle ADO$, one finds $OD = 8\sqrt{2}$. The volume of the octahedron = the volume of two pyramids, etc.

38. Find the total area and the volume of a regular octahedron whose edge is 18 in.

39. The altitude of a regular pyramid is 16 in. and each side of the square base is 24 in. Find the lateral area and the volume.

40. The slant height of a regular pyramid is 16 in. and each face is an equilateral triangle whose side is $20\sqrt{3}$ in. Find the total area and the volume.

41. The altitude of a regular pyramid is 29 in. and its base is a regular hexagon whose side is 10 in. Find the total area and the volume.

42. Find the total area and the volume of a regular tetrahedron whose edge is 18 in.

43. Find the total area and the volume of a regular octahedron whose edge is 20 in.

44. If the edge of a regular tetrahedron is e in., show that the total area is $e^2\sqrt{3}$ in. and the volume is $\frac{1}{12}\, e^3\sqrt{2}$ cu. in.

45. If the edge of a regular octahedron is e in., show that the total area is $2\, e^2\sqrt{3}$ sq. in. and the volume is $\frac{1}{3}\, e^3\sqrt{2}$ cu. in.

46. A pyramid whose base is a square 9 in. on a side, contains 360 cu. in. Find its height.

47. A pyramid has for its base a hexagon whose side is $7\frac{1}{2}$ units and the pyramid contains 675 cubic units. Find the altitude.

48. The volume of a regular tetrahedron is $144\sqrt{2}$ cu. in. Find its edge.

49. The volume of a regular octahedron is $243\sqrt{2}$ cu. in. Find its edge.

50. The volume of a square pyramid is 676 cu. in. and the altitude is 1 ft. Find the side of the base. Find the lateral area.

51. The altitude of the Great Pyramid is 480 ft. and its base is 764 ft. square. It is said to have cost \$10 a cubic yard and \$3 more for each square yard of lateral surface (considered as planes). What was the cost?

52. The total surface of a regular tetrahedron is $324\sqrt{3}$ sq. in. Find its volume.

53. The base of a pyramid is an isosceles right triangle whose hypotenuse is 8 in. The altitude of the pyramid is 15 in. Find the volume.

54. Find the area of the section of a triangular pyramid, each side of whose base is 8 in. and whose altitude is 18 in., made by a plane parallel to the base and 1 ft. from the vertex.

55. The altitude of a frustum of a pyramid is 6 in., and the areas of the bases are 20 sq. in. and 45 sq. in. Find the altitude of the complete pyramid. Find the volume of this frustum by two distinct methods.

56. A granite monument in the form of a frustum of a pyramid, having rectangular bases one of which is 8 ft. wide and 12 ft. long, and the other 6 ft. wide, is 30 ft. high. It is surmounted by a granite pyramid having the same base as the less base of the frustum, and 10 ft. in height. Find the entire volume and the weight. [1 cu. ft. of water weighs $62\frac{1}{2}$ lb. and granite is 3 times as heavy as water.]

57. If a square pyramid contains 40 cu. in. and its altitude is 15 in., find the side of its base.

58. A church spire in the form of a regular hexagonal pyramid whose base edge is 8 ft. and whose altitude is 75 ft. is to be painted at the rate of 18 ¢ per square yard. Find the cost.

59. Find the edge of a cube whose volume is equal to the volumes of two cubes whose edges are 4 in. and 6 in.

60. The base of a certain pyramid is an isosceles trapezoid whose parallel sides are 20 ft. and 30 ft. and the equal sides each 13 ft. Find the volume of the pyramid if its altitude is 12 yd.

61. The lateral edge of the frustum of a regular square pyramid is 53 in. and the sides of the bases are 10 in. and 66 in. Find the altitude, the slant height, the lateral area, and the volume.

62. The sides of the base of a triangular pyramid in inches are 33, 34, 65, and the altitude of the pyramid is 80. Find its volume.

63. The sides of the base of a tetrahedron in inches are 17, 25, 26, and its altitude is 90. Find its volume.

64. If there are $1\frac{1}{4}$ cu. ft. in a bushel, what is the capacity (in bushels) of a hopper in the shape of an inverted pyramid, 12 ft. deep and 8 ft. square at the top?

65. In the corner of a cellar is a pyramidal heap of coal. The base of the heap is an isosceles right triangle whose hypotenuse is 20 ft. and the altitude of the heap is 7 ft. If there are 35 cu. ft. in a ton of coal, how many tons are there in this heap?

66. How many cubic yards of earth must be removed in digging an artificial lake 15 ft. deep, whose base is a rectangle 180 ft. × 20 ft. and whose top is a rectangle 216 ft. × 24 ft.? [The frustum of a pyramid.]

67. One pair of homologous edges of two similar tetrahedrons are 3 ft. and 5 ft. Find the ratio of their surfaces; of their volumes.

68. A pair of homologous edges of two similar polyhedrons are 5 in. and 7 in. Find the ratio of their surfaces; of their volumes.

69. The edge of a cube is 3 in. What is the edge of a cube twice as large? four times as large? half as large?

70. An edge of a tetrahedron is 6 ft. What is the edge of a similar tetrahedron three times as large? eight times as large? nine times as large? one third as large?

71. An edge of a regular icosahedron is 3 in. What is the edge of a similar solid five times as large? ten times as large? fifty times as large? a thousand times as large?

72. The edges of a trunk are 2 ft., 3 ft., 5 ft. Another trunk is twice as long (the other edges 2 ft. $\times$ 3 ft.). How do their volumes compare? A third trunk has each dimension double those of the first. How does its volume compare with the first? How do their surfaces compare?

73. If the altitude of a certain regular pyramid is doubled, but the base remains unchanged, how is the volume affected? If each edge of the base is doubled, but the altitude is unchanged, how is the volume affected? If the altitude and each edge of the base are all doubled, how is the volume affected?

74. A contractor agrees to build a dam 60 ft. long, 15 ft. high, 11 ft. wide at the bottom and 7 ft. wide at the top for \$8.25 a cubic yard. Find his profit if it costs him only \$2000.

75. A pyramid is cut by a plane parallel to the base and bisecting the altitude. What part of the entire pyramid is the less pyramid cut away by this plane?

76. The volume of a certain pyramid, one of whose edges is 7 in., is 686 cu. in. Find the volume of a similar pyramid whose homologous edge is 8 in.

77. A certain polyhedron whose shortest edge is 2 in. weighs 40 lb. What is the weight of a similar polyhedron whose shortest edge is 5 in.?

78. An edge of a polyhedron is 5 in. and the homologous edge of a similar polyhedron is 7 in. The entire surface of the first is 250 sq. in. and its volume is 375 cu. in. Find the entire surface and volume of the second.

79. A berry box, said to contain a quart of berries, is in the form of the frustum of a pyramid 5 in. square at the top, $4\frac{1}{2}$ in. square at the bottom, and $2\frac{7}{8}$ in. deep. If a U.S. dry quart contains 67.2 cu. in., does this box contain more or less than a quart?

BOOK VIII

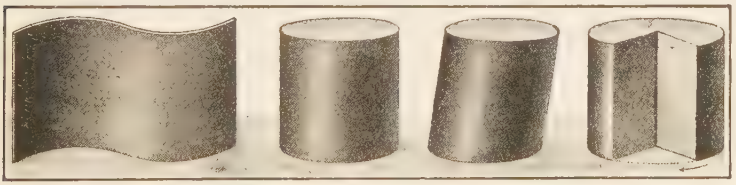
CYLINDERS AND CONES

CYLINDERS

638. A **cylindrical surface** is a surface generated by a moving straight line which continually intersects a given curved line in a plane, and which is always parallel to a given straight line not in the plane of the curve.

The generating line is the **generatrix.** The directing curve is the **directrix.**

An **element** of a cylindrical surface is the generating line in any position.



CYLINDRICAL SURFACE | RIGHT CIRCULAR CYLINDER | OBLIQUE CYLINDER | CYLINDER OF REVOLUTION

639. A **cylinder** is a solid bounded by a cylindrical surface and two parallel planes.

The **bases** of a cylinder are the parallel plane surfaces.

The **lateral area** of a cylinder is the area of the cylindrical surface included between the planes of the bases.

The **total area** of a cylinder is the sum of the lateral area and the areas of the bases.

The **altitude** of a cylinder is the perpendicular distance between the planes of the bases.

640. A **right cylinder** is a cylinder whose elements are perpendicular to the planes of the bases.

A **circular cylinder** is a cylinder whose base is a circle.

An **oblique cylinder** is a cylinder whose elements are not perpendicular to the planes of the bases.

A **right circular cylinder** is a right cylinder whose base is a circle.

A **cylinder of revolution** is a cylinder generated by the revolution of a rectangle about one of its sides as an axis.

Similar cylinders of revolution are cylinders generated by similar rectangles revolving on homologous sides.

641. A **right section** of a cylinder is a section made by a plane perpendicular to all the elements.

A **plane** is **tangent** to a cylinder if it contains one element of the cylindrical surface and only one, however far it may be extended.

A **prism** is **inscribed** in a cylinder if its lateral edges are elements of the cylinder and the bases of the prism are inscribed in the bases of the cylinder.

A **prism** is **circumscribed** about a cylinder if its lateral faces are tangent to the cylinder and the bases of the prism are circumscribed about the bases of the cylinder.

PRELIMINARY THEOREMS

642. THEOREM. **Any two elements of a cylinder are parallel and equal.** (495 and 509.)

643. THEOREM. **A line drawn through any point in a cylindrical surface, parallel to an element, is itself an element.** (Ax. 13.)

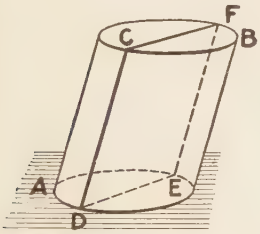
644. THEOREM. **A right circular cylinder is a cylinder of revolution.**

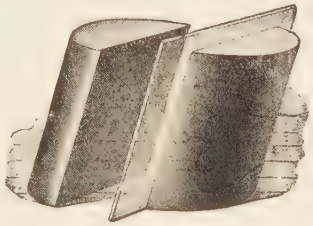
Ex. If a plane is defined as a surface generated by a moving straight line, what is the directrix?

THEOREMS AND DEMONSTRATIONS

Proposition I. Theorem

645. Every section of a cylinder made by a plane containing an element is a parallelogram.





Given: Cylinder AB; plane CE containing element CD.

To Prove: CE is a $\square$.

Proof: At E draw $EF \parallel$ to CD in plane CE. Also, EF is an element of the cylinder. (643.)

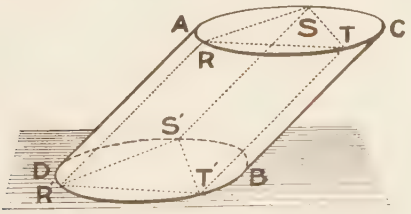
$\therefore$ EF is the intersection of the plane and the cylindrical surface. (466.)

Also CF is $\parallel$ to DE (484).

$\therefore$ $CDEF$ is a $\square$ (120).

Proposition II. Theorem

646. The bases of a cylinder are congruent.

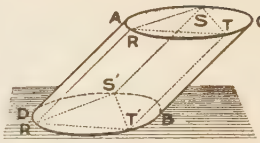


Given: (?). **To Prove**: (?).

Proof: Suppose R, S, T three points in the perim. of base AC.

Draw elements RR', SS', TT'.

Also draw RS, ST, RT, $R'S'$, $S'T'$, $R'T'$.



RR' = and is $\parallel$ to SS'
RR' = and is $\parallel$ to TT' (642).
SS' = and is $\parallel$ to TT'

$\therefore$ RS' is a $\square$, RT' is a $\square$, ST' is a $\square$ (129).

$\therefore$ $RS = R'S'$; $ST = S'T'$; $RT = R'T'$ (124).

$\triangle RST \cong \triangle R'S'T'$ (?).

$\therefore$ base AC may be placed upon base BD so that R, S, and T coincide with R', S', and T', respectively. But S is *any point* on the boundary; hence *every point* on the boundary of AC will coincide with a corresponding point on the boundary of BD.

$\therefore$ base $AC \cong$ base BD (Def. 26).

Q.E.D.

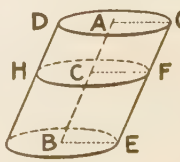
647. COROLLARY. **Parallel plane sections of a cylinder (cutting all the elements) are congruent.**

648. COROLLARY. **Every section of a right cylinder made by a plane containing an element is a rectangle.**

649. COROLLARY. **The line joining the centers of the bases of a circular cylinder is equal and parallel to an element.**

650. COROLLARY. **All sections of a circular cylinder parallel to its bases are equal circles, and the straight line joining the centers of the bases passes through the centers of all the parallel sections.**

To Prove: HF = any other $\parallel$ section and C is its center.



Proof: Pass a plane AE cutting the three planes in AG, CF, and BE.

DF is a cylinder (Def. 639).

$\therefore$ $HF \cong DG$ (646).

That is, HF is a $\odot$ and equal to any other section $\parallel$ to DG.

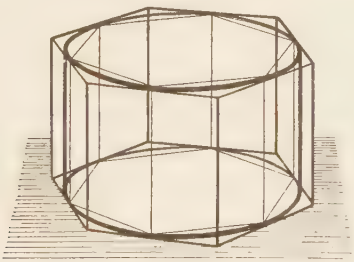
Now	AG is $\parallel$ to CF	(484).
And	AB is $\parallel$ to GE	(649).
	$\therefore$ AF is a $\square$	(120).
And	$AG = CF$	(124).

But HF is a $\odot$ and F is any point on it. Hence C is equally distant from all points on HF, and is, therefore, the center. (179.)

Q.E.D.

Proposition III. Theorem

651. Theorem. **If a regular prism is inscribed in, or circumscribed about, a right circular cylinder and the number of sides of the base is indefinitely increased, the lateral area of the cylinder is the limit of the lateral area of the prism.**



Given: A regular prism inscribed in and a regular prism circumscribed about a right circular cylinder; the lateral area of the cylinder $= L$, and of the prisms, L_i and L_c, respectively.

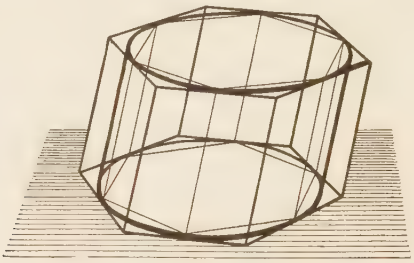
To Prove: That as the number of sides of the bases of the prisms is indefinitely increased, L is the limit of both L_i and L_c.

Proof: If the number of sides of the bases of the prisms is indefinitely increased, their perimeters will approach the circumference of the base of the cylinder as a limit. (?.)

Hence, it is obvious that the lateral area of the cylinder is the limit of the lateral area of either prism. Q.E.D.

Proposition IV. Theorem

652. If a prism having a regular polygon for a base is inscribed in, or circumscribed about, any circular cylinder and the number of the sides of the base of the prism is indefinitely increased, the volume of the cylinder is the limit of the volume of the prism.



Given: (?). **To Prove:** (?).

Proof: If the number of sides of the base of either prism is indefinitely increased, the area of the base of the prism approaches the area of the base of the cylinder. (424, II.)

∴ it is obvious that the volume of the cylinder is the limit of the volume of either prism. Q.E.D.

Ex. 1. If the cylindrical surface of a cylinder is cut along an element, and this surface is placed in coincidence with a plane, what plane geometrical figure will it become?

Ex. 2. What two lines determine the size of a right circular cylinder?

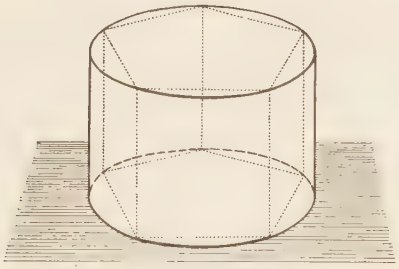
Ex. 3. What is the locus of all points 2 in. from a circular cylindrical surface? Is the answer to this question affected by the radius of the given surface? If so, explain.

Ex. 4. From a log 36 in. in diameter at its less end, is to be cut the largest prismatic piece of timber possible, having square ends. Find the side of this square.

Ex. 5. A lead pencil whose ends are regular hexagons was cut from a cylindrical piece of wood, with the least waste of wood. If the original piece was 8 in. long and $\frac{1}{2}$ in. in diameter, find the volume of the pencil.

Proposition V. Theorem

653. The lateral area of a right circular cylinder is equal to the product of the circumference of the base by an element.



Given: A right circular cylinder, the circumference of whose base $= C$, and whose element $= E$.

To Prove: Lateral area $L = C \cdot E$.

Proof: Inscribe in the cylinder a regular prism, the perimeter of whose base is P, whose lateral edge is E, and whose lateral area is L'.

Then $L' = P \cdot E$ (?).

If the number of sides of the base of the prism is indefinitely increased, L' approaches L as a limit (?).

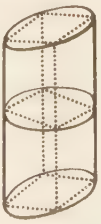
P approaches C as a limit (?).

$\therefore L = C \cdot E$ (?). Q.E.D.

654. Corollary. Area of a right circular cylinder:

$$L = 2\pi RH. \qquad T = 2\pi RH + 2\pi R^2 = 2\pi R(H + R).$$

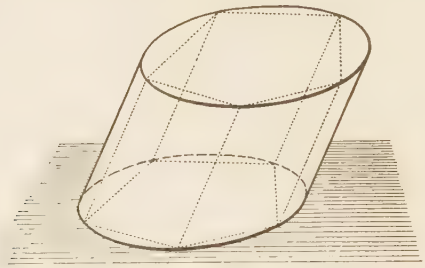
(Where L = lateral area, H = altitude, R = radius of base, and T = total area.)



Note. The lateral area of an oblique circular cylinder equals the product of the perimeter of a right section of the cylinder by an element. The right section of an oblique circular cylinder is not a circle. The right section of an inscribed prism, having a regular polygon for a base, is not a regular polygon. Hence the proof of this theorem is omitted.

Proposition VI. Theorem

655. The volume of a circular cylinder is equal to the product of its base by its altitude.



Given: A circular cylinder whose base $= B$, altitude $= H$, and volume $= V$.

To Prove: $V = B \cdot H$.

Proof: Inscribe a prism having a regular polygon for its base, whose base $= B'$ and volume $= V'$.

Then $V' = B' \cdot H$ (?).

If the number of sides of the base of the prism is indefinitely increased, V' approaches V as a limit (?).

B' approaches B as a limit (?).

$B' \cdot H$ approaches $B \cdot H$ as a limit.

$\therefore V = B \cdot H$ (229). Q.E.D.

656. Corollary. Volume of a circular cylinder:

$$V = \pi R^2 H.$$

(Where V = volume, H = altitude, and R = radius of base.)

Ex. 1. Find the lateral area and the total area of a cylinder whose altitude is 15 in. and radius 14 in.

Ex. 2. How many square inches of tin are required to make a cylindrical pipe $10\frac{1}{2}$ in. in diameter and 8 ft. long?

Ex. 3. What is the capacity of a cylindrical pail 1 ft. high and 9 in. in diameter?

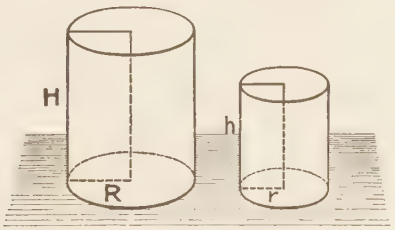
Proposition VII. Theorem

657. Of two similar cylinders of revolution:

I. The lateral areas are to each other as the squares of their altitudes or as the squares of the radii of their bases.

II. The total areas are to each other as the squares of their altitudes or as the squares of the radii of their bases.

III. The volumes are to each other as the cubes of their altitudes or as the cubes of the radii of their bases.



Given: Two similar cylinders of revolution whose lateral areas $= L$ and l; total areas $= T$ and t; volumes $= V$ and v; altitudes $= H$ and h, and radii are R and r.

To Prove: I. $L : l = H^2 : h^2 = R^2 : r^2$.

II. $T : t = H^2 : h^2 = R^2 : r^2$.

III. $V : v = H^3 : h^3 = R^3 : r^3$.

Proof: The generating rectangles are similar. (Def. 640.)

$$\therefore H : h = R : r \qquad (?).$$

Hence
$$\frac{H+R}{h+r} = \frac{H}{h} = \frac{R}{r} \qquad (291).$$

I. $$\frac{L}{l} = \frac{2\pi RH}{2\pi rh} = \frac{RH}{rh} = \frac{R}{r} \cdot \frac{H}{h} = \frac{H}{h} \cdot \frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2} \qquad (Ax.\ 6).$$

II. $$\frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R}{r} \cdot \frac{H+R}{h+r} = \frac{H}{h} \cdot \frac{H}{h} = \frac{H^2}{h^2} = \frac{R^2}{r^2} \qquad (Ax.\ 6).$$

III. $$\frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2 H}{r^2 h} = \frac{R^2}{r^2} \cdot \frac{H}{h} = \frac{H^2}{h^2} \cdot \frac{H}{h} = \frac{H^3}{h^3} = \frac{R^3}{r^3} \qquad (Ax.\ 6).$$

Q.E.D.

ORIGINAL EXERCISES (NUMERICAL)

$\pi = 3\frac{1}{7}$. 1 bu. = 2150.42 cu. in. 1 gal. = 231 cu. in.

In a cylinder of revolution:

1. If $R = 5$ in., $H = 14$ in., find L; T; V.
2. If $R = 7$ m., $H = 10$ m., find L; T; V.
3. If $R = 4\frac{2}{3}$ ft., $H = 18$ ft., find L; T; V.
4. If $R = 6$ in., $L = 792$ sq. in., find H; T; V.
5. If $R = 4$ ft., $T = 352$ sq. ft., find H; L; V.
6. If $R = 2$ in., $V = 22$ cu. in., find H; L; T.
7. If $H = 5.6$ in., $L = 352$ sq. in., find R; T; V.
8. If $H = 9$ in., $T = 440$ sq. in., find R; L; V.
9. If $H = 9\frac{1}{3}$ in., $V = 66$ cu. in., find R; L; T.

10. How many square inches of tin will be required to make a cylindrical pail 10 in. in diameter and 1 ft. in height, without any lid? How many gallons will it contain?

11. The diameter of a cylindrical well is $5\frac{1}{2}$ ft. and the water is 14 ft. deep. How many gallons of water does the well hold?

12. In a cylinder of revolution generated by a rectangle 30 in. × 14 in. revolving about its shorter side as an axis, find L; T; V.

13. In a cylinder of revolution generated by the rectangle of No. 12, revolving about its longer side as an axis, find L; T; V.

14. A cylindrical vessel 9 in. high, closed at one end, required $361\frac{3}{7}$ sq. in. of tin in its construction. Find its radius.

15. A cylindrical pail 12 in. high holds exactly 2 gal. Find R.

16. How many cubic feet of metal are there in a hollow cylindrical tube 42 ft. long, whose outer and inner diameters are 10 in. and 6 in.?

17. A tunnel whose cross section is a semicircle 18 ft. high is 1 mi. long. How many cubic yards of material were removed in the excavation?

18. An irregular stone is placed in a cylindrical vessel a in. in diameter and partly full of water. The water rises b in. Find volume of stone.

19. A rod of copper 18 ft. long and 2 in. square at the end is melted and formed into a wire $\frac{1}{8}$ in. in diameter. Find the length of the wire.

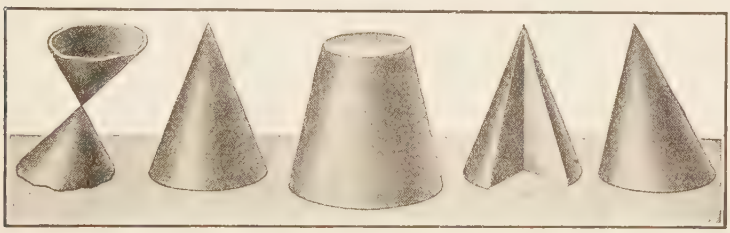
20. How many miles of platinum wire $\frac{1}{32}$ in. in diameter can be made from a cubic foot of platinum?

21. If a cubic foot of copper weighs 550 lb., what is the weight of a copper wire $\frac{1}{8}$ in. in diameter and 5 mi. long?

CONES

658. A **conical surface** is a surface generated by a moving straight line that continually intersects a given curve in a plane, and passes through a fixed point not in this plane.

The generating line is the **generatrix.** The directing curve is the **directrix.** The fixed point is the **vertex** of the conical surface. An **element** of a conical surface is the generating line in any position.



CONICAL SURFACE | RIGHT CIRCULAR CONE | FRUSTUM OF A CONE | CONE OF REVOLUTION | OBLIQUE CONE

659. A **cone** is a solid bounded by a conical surface and a plane cutting all the elements.

The **base** of a cone is its plane surface.

The **lateral area** of a cone is the area of the conical surface.

The **total area** of a cone is the sum of the lateral area and the area of the base.

The **altitude** of a cone is the perpendicular distance from the vertex to the plane of the base.

660. A **circular cone** is a cone whose base is a circle.

The **axis** of a circular cone is the line drawn from the vertex to the center of the base.

A **right circular cone** is a circular cone whose axis is perpendicular to the plane of the base.

An **oblique circular cone** is one whose axis is oblique to the plane of the base.

A **cone of revolution** is a cone generated by the revolution of a right triangle about one of the legs as an axis.

Similar cones of revolution are cones generated by the revolution of similar right triangles revolving about homologous sides.

The **slant height** of a cone of revolution is any one of its elements.

661. A **frustum** of a cone is the portion of a cone between the base and a plane parallel to the base.

The **altitude** of a frustum of a cone is the perpendicular distance between the planes of its bases. The **slant height** of a frustum of a cone is the portion of an element included between the bases. The **lateral area** of a frustum is the area of its curved surface. The **total area** of a frustum is the sum of the lateral area and the area of the bases. The **mid-section** of a frustum is the section made by a plane parallel to the bases and bisecting the altitude and the slant height.

662. A **plane** is **tangent** to a cone if it contains one element of the conical surface and only one, however far it may be extended.

A **pyramid** is **inscribed** in a cone if its base is inscribed in the base of the cone, and its vertex is the vertex of the cone.

A **pyramid** is **circumscribed** about a cone if its base is circumscribed about the base of the cone, and its vertex is the vertex of the cone.

The **frustum** of a pyramid is **inscribed** in, or **circumscribed** about, the frustum of a cone if the bases of the former are inscribed in, or circumscribed about, the bases of the latter.

Ex. 1. Find the slant height of a right circular cone whose altitude is 8 in. and whose radius is 6 in.

Ex. 2. What is the locus of all points 3 in. from a conical surface?

Ex. 3. What is the locus of all lines forming a given angle with a given line at a given point in the line?

PRELIMINARY THEOREMS

663. THEOREM. The elements of a right circular cone are all equal. (504, II.)

664. THEOREM. A right circular cone is a cone of revolution.

665. THEOREM. The altitude of a cone of revolution is the axis of the cone.

666. THEOREM. A straight line drawn from the vertex of a cone to any point in the perimeter of the base is an element. (39.)

667. THEOREM. The lateral edges of a pyramid inscribed in a cone are elements of the cone.

668. THEOREM. The lateral faces of a pyramid circumscribed about a cone are tangent to the conical surface.

669. THEOREM. The slant height of a regular pyramid circumscribed about a right circular cone is the same as the slant height of the cone.

670. THEOREM. The slant height of the frustum of a regular pyramid circumscribed about the frustum of a right circular cone is the same as the slant height of the frustum of the cone.

671. THEOREM. The radius of the mid-section of a frustum of a right circular cone is equal to half the sum of the radii of the bases.

Proof: The radius of the mid-section is the median of a trapezoid whose bases are the radii of the bases of the frustum. That is, $m = \frac{1}{2}(R+r)$. (139.)

Q.E.D.

Ex. If a conical surface is cut along an element and the surface there placed in coincidence with a plane, what geometrical figure does the surface become?

672. THEOREM. If a regular pyramid is inscribed in, or circumscribed about, a right circular cone and the number of sides of the base is indefinitely increased, the lateral area of the cone is the limit of the lateral area of the pyramid. (See Fig. A.)

Demonstration is similar to that of 651.

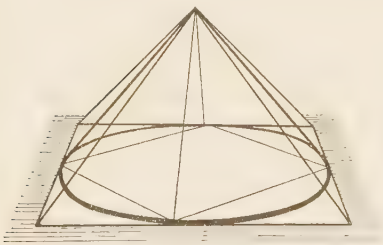


FIG. A

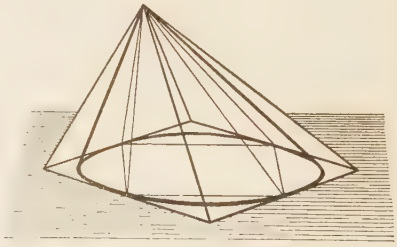


FIG. B

673. THEOREM. If a pyramid having a regular polygon for a base is inscribed in, or circumscribed about, any circular cone and the number of sides of its base is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid. (See Fig. B.)

Demonstration is similar to that of 652.

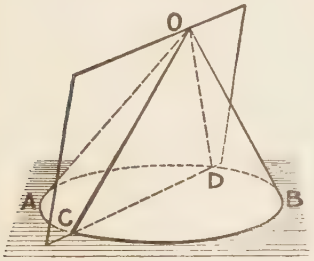
674. THEOREM. If a frustum of a regular pyramid is inscribed in, or circumscribed about, the frustum of a right circular cone and the number of sides of the bases is indefinitely increased, the lateral area of the frustum of the cone is the limit of the lateral area of the frustum of the pyramid.

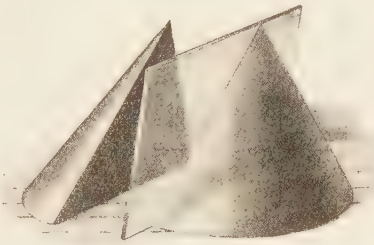
675. THEOREM. If the frustum of a pyramid having regular polygons for its bases is inscribed in, or circumscribed about, a frustum of any circular cone and the number of sides of the bases of the frustum is indefinitely increased, the volume of the frustum of the cone is the limit of the volume of the frustum of the pyramid.

THEOREMS AND DEMONSTRATIONS

Proposition VIII. Theorem

676. Any section of a cone made by a plane passing through the vertex is a triangle.





Given: Cone *O–AB*; plane *OCD*.

To Prove: Section *OCD* is a △.

Proof: Draw straight lines *OC*, *OD*, in plane *OCD*.

They are elements (666).

∴ *OC* and *OD* compose the intersection of the plane and the conical surface. (466.)

Also *CD* is a straight line (?).

∴ *OCD* is a △ (23).

Q.E.D.

Ex. 1. Can the plane section of a cone, containing the vertex, ever be a right triangle? Explain. Can it be an isosceles triangle? Explain. What kind of triangle is this section, in general?

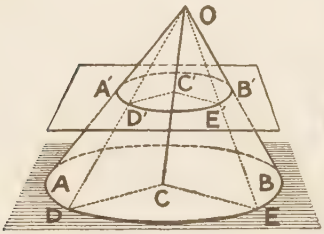
Ex. 2. What is the locus of all straight lines making a given angle with a given plane at a given point in the plane?

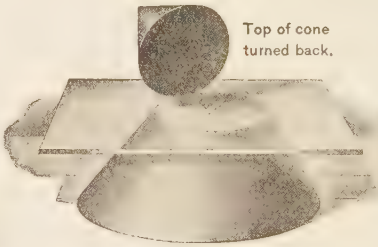
Ex. 3. Does every cone have a slant height, or only certain kinds of cones?

Ex. 4. If a circular disk is held between a source of light and a wall, and parallel to the wall, what is the shape of the shadow on the wall? What is the shape of the shadow region between the light and the wall? (Consider the source of light a point.)

Proposition IX. Theorem

677. Any section of a circular cone made by a plane parallel to the base is a circle, whose center is the intersection of the plane with the axis.





Given: Cone O–AB; circle C its base; section $A'B' \parallel$ to base, and axis OC.

To Prove: $A'B'$ also a $\odot$, whose center is C'.

Proof: Pass planes OCD, OCE intersecting the base in CD, CE respectively, and the section in $C'D'$, $C'E'$.

In $\triangle OCD$ and OCE, $D'C'$ is $\parallel$ to DC;

$C'E'$ is $\parallel$ to CE (484).

$\therefore \triangle OC'D'$ is similar to $\triangle OCD$;

$\triangle OC'E'$ is similar to $\triangle OCE$ (305).

$$\therefore \frac{OC'}{OC} = \frac{C'D'}{CD} \text{ and } \frac{OC'}{OC} = \frac{C'E'}{CE} \quad (313).$$

$$\therefore \frac{C'D'}{CD} = \frac{C'E'}{CE} \quad (\text{Ax. } 1).$$

But $CD = CE$ (187).

$\therefore$ by multiplying, $C'D' = C'E'$ (Ax. 3).

That is, all points on the boundary of $A'B'$ are equally distant from C'.

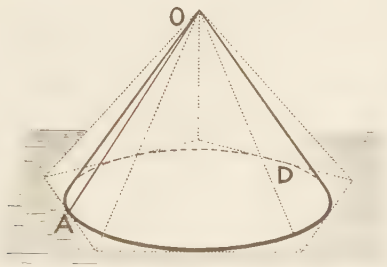
$\therefore A'B'$ is a $\odot$ whose center is C (179). Q.E.D.

Ex. Any section of a circular cone parallel to the base is to the base as the square of its distance from the vertex is to the altitude of the cone.

Proof: $A'B' : AB = \overline{C'D'}^2 : \overline{CD}^2 = \overline{OC'}^2 : \overline{OC}^2$. (Explain.)

Proposition X. Theorem

678. **The lateral area of a right circular cone is equal to half the product of the circumference of the base by the slant height.**



Given: Right circular cone $O\text{-}AD$, the circumference of whose base $= C$ and whose slant height $= s$.

To Prove: Lateral area $= \frac{1}{2} C \cdot s$.

Proof: Circumscribe a regular pyramid and denote the lateral area by L' and the perimeter of the base by P.

Slant height $OA = s$. (663.)

Then $L' = \frac{1}{2} P \cdot s$ (598).

Now indefinitely increase the number of sides of the base of the pyramid and L' approaches L as a limit (672),

P approaches C as a limit (424, I),

$\frac{1}{2} P \cdot s$ approaches $\frac{1}{2} C \cdot s$ as a limit (?).

Hence $L = \frac{1}{2} C \cdot s$ (229). Q.E.D.

679. Corollary. **Area of a right circular cone:**

$$L = \frac{1}{2}(2\pi R)s = \pi Rs. \qquad T = \pi Rs + \pi R^2 = \pi R(s + R).$$

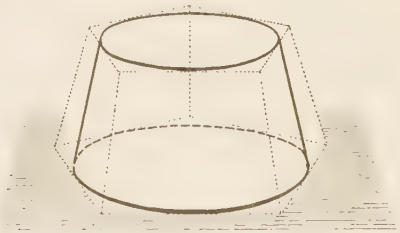
(Where L = lateral area, T = total area, s = slant height, and R = radius of base.)

Ex. 1. Of a right circular cone whose slant height is 15 in. and radius is 9 in., find the lateral area, the total area, and the altitude.

Ex. 2. If the radius of a right circular cone is 8 in. and the altitude is 15 in., find the slant height, the lateral area, and the total area.

Proposition XI. Theorem

680. The lateral area of the frustum of a right circular cone is equal to half the sum of the circumferences of the bases multiplied by the slant height.



Given: Frustum of right circular cone, whose lateral area is L; whose slant height is s; and the circumferences of whose bases are C and c.

To Prove: $L = \frac{1}{2}(C + c) \cdot s$.

Proof: Circumscribe a frustum of a regular pyramid and denote its lateral area by L', the perimeters of its bases by P and p. The slant height of frustum of pyramid $= s$ (670).

Now $L' = \frac{1}{2}(P + p) \cdot s$ (605).

Indefinitely increase the number of the sides of the bases of the frustum of the pyramid and L' approaches L as a limit, P approaches C, and p approaches c (?).

$\frac{1}{2}(P + p) \cdot s$ approaches $\frac{1}{2}(C + c) \cdot s$ as a limit.

Hence $L = \frac{1}{2}(C + c) \cdot s$ (229). Q.E.D.

681. Corollary. **Area of the frustum of a right circular cone:**

$$L = \tfrac{1}{2}(2\pi R + 2\pi r)s = \pi(R + r)s.$$

$$L = \pi(2m)s = 2\pi ms.$$

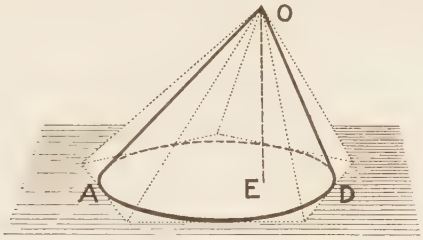
$$T = \pi(R + r)s + \pi R^2 + \pi r^2.$$

$$\therefore T = \pi[(R + r)s + R^2 + r^2].$$

(Where L, T, R, r and s denote magnitudes as before, and m = the radius of the midsection.)

Proposition XII. Theorem

682. The volume of a circular cone is equal to one third the product of the area of the base by the altitude.



Given: Circular cone O–AD, whose volume $= V$; area of whose base $= B$; altitude $= OE = H$.

To Prove: $V = \frac{1}{3} B \cdot H$.

Proof: Circumscribe (or inscribe) a pyramid having a regular polygon for its base. Denote the volume of the pyramid by V', its base by B'. Its altitude $= OE = H$ (491).

$\therefore V' = \frac{1}{3} B' \cdot H$ (?).

Indefinitely increase the number of the sides, etc.

Then V' approaches V as a limit (?).

B' approaches B as a limit (424, II).

$\frac{1}{3} B' \cdot H$ approaches $\frac{1}{3} B \cdot H$ as a limit.

Hence $V = \frac{1}{3} B \cdot H$ (229). Q.E.D.

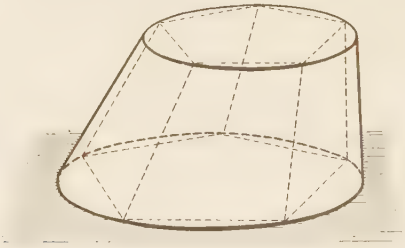
683. Corollary. Volume of a circular cone: $V = \frac{1}{3} \pi R^2 H$.
(Where V = volume, H = altitude, and R = radius of base.)

Ex. 1. Find the volume of a circular cone whose altitude is 14 in. and the radius of the base, 6 in.

Ex. 2. A right triangle, whose legs are 15 in. and 20 in., is revolved about the lesser leg as an axis, forming a cone of revolution. Find the lateral area, the total area, and the volume. Find the lateral area, the total area, and the volume of the cone formed by revolving this triangle about the greater leg as an axis.

Proposition XIII. Theorem

684. The volume of the frustum of a circular cone is equal to one third the product of the altitude by the sum of the lower base, the upper base, and a mean proportional between the bases of the frustum.



Given: A frustum of any circular cone, whose volume $= V$, whose bases are B and b, whose altitude is H.

To Prove: $V = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Proof: Inscribe (or circumscribe) a frustum of a pyramid having regular polygons for bases.

Denote its volume by V', bases by B' and b', and altitude by H.

Then $V' = \frac{1}{3} H[B' + b' + \sqrt{B' \cdot b'}]$ (618).

Indefinitely increase, etc. V' approaches V as a limit (?),

B' and b' approach B and b respectively as limits (?),

$\sqrt{B' \cdot b'}$ approaches $\sqrt{B \cdot b}$ as a limit.

$\frac{1}{3} H[B' + b' + \sqrt{B' \cdot b'}]$ approaches $\frac{1}{3} H[B + b + \sqrt{B \cdot b}]$.

Hence $V = \frac{1}{3} H[B + b + \sqrt{B \cdot b}]$ (229). Q.E.D.

685. Corollary. Volume of the frustum of a circular cone,

$$V = \tfrac{1}{3} H[\pi R^2 + \pi r^2 + \sqrt{\pi R^2 \cdot \pi r^2}]$$

$$\therefore \boldsymbol{V = \tfrac{1}{3} \pi H [R^2 + r^2 + R \cdot r]}.$$

(Where $V =$ volume, etc.)

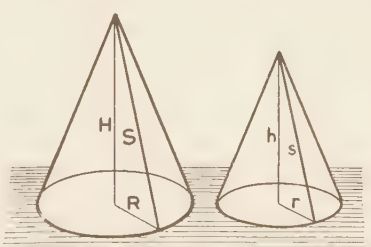
Proposition XIV. Theorem

686. Of two similar cones of revolution:

I. The lateral areas are to each other as the squares of their altitudes, or as the squares of their radii, or as the squares of their slant heights.

II. The total areas are to each other as the squares of their altitudes, or as the squares of their radii, or as the squares of their slant heights.

III. The volumes are to each other as the cubes of their altitudes, or as the cubes of their radii, or as the cubes of their slant heights.



Given: Two similar cones of revolution, whose respective lateral areas are L and l, total areas are T and t, volumes are V and v, altitudes are H and h, radii are R and r, slant heights are S and s.

To Prove: I. $\frac{L}{l}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$. II. $\frac{T}{t}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$.

III. $\frac{V}{v}=\frac{H^3}{h^3}=\frac{R^3}{r^3}=\frac{S^3}{s^3}$.

Proof: Generating $\triangle$ are similar and $\frac{H}{h}=\frac{R}{r}=\frac{S}{s}$ (?).

Hence $\frac{R+S}{r+s}=\frac{R}{r}=\frac{S}{s}=\frac{H}{h}$ (291).

I. $\frac{L}{l}=\frac{\pi RS}{\pi rs}=\frac{R}{r}\cdot\frac{S}{s}=\frac{H}{h}\cdot\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$ (Ax. 6).

II. $\frac{T}{t}=\frac{\pi R(R+S)}{\pi r(r+s)}=\frac{R}{r}\cdot\frac{R+S}{r+s}=\frac{H}{h}\cdot\frac{H}{h}=\frac{H^2}{h^2}=\frac{R^2}{r^2}=\frac{S^2}{s^2}$ (Ax. 6).

III. $\frac{V}{v}=\frac{\frac{1}{3}\pi R^2H}{\frac{1}{3}\pi r^2h}=\frac{R^2}{r^2}\cdot\frac{H}{h}=\frac{H^2}{h^2}\cdot\frac{H}{h}=\frac{H^3}{h^3}=\frac{R^3}{r^3}=\frac{S^3}{s^3}$ (?). Q.E.D.

ORIGINAL EXERCISES

In a cone of revolution:

1.	If $H = 12$ in.,	$s = 13$ in.,	find R.
2.	If $H = 15$ ft.,	$R = 8$ ft.,	find s.
3.	If $R = 18$ cm.,	$s = 30$ cm.,	find H.
4.	If $H = 6$ in.,	$s = 10$ in.,	find R; L; T; V.
5.	If $H = 20$ ft.,	$R = 21$ ft.,	find s; L; T; V.
6.	If $R = 7$ m.,	$s = 25$ m.,	find H; L; T; V.
7.	If $L = 4070$ sq. in.,	$s = 37$ in.,	find R; H; T; V.
8.	If $L = 46.64$ sq. in.,	$R = 2.8$ in.,	find s; H; T; V.
9.	If $L = 400$ sq. ft.,	$T = 500$ sq. ft.,	find s; H; R; V.
10.	If $T = 80\pi$ sq. in.,	$R = 5$ in.,	find s; H; L; V.
11.	If $T = 10\pi$ sq. ft.,	$s = 3$ ft.,	find R; H; L; V.
12.	If $V = 462$ cu. in.,	$R = 21$ in.,	find H; s; L; T.
13.	If $V = 8\frac{8}{175}$ cu. ft.,	$H = 3$ ft.,	find R; s; L; T.

14. What would be the cost at 10¢ a square foot of painting a conical church steeple, 112 ft. high and 30 ft. in diameter at the base?

15. The sides of an equilateral triangle are each 12 in. Find the lateral surface, the total surface, and the volume of the solid generated by revolving this triangle about an altitude as an axis.

16. An isosceles right triangle whose legs are each 8 in. is revolved about the hypotenuse as an axis. Find the total surface and the volume of the solid generated.

17. The sides of an equilateral triangle are each 10 in. Find the total surface and the volume of the solid generated by revolving this triangle about one of its sides as an axis.

18. Find the volume of a cone of revolution whose slant height is 16 in. and whose lateral area is 192π sq. in.

19. Find the lateral area of a cone of revolution whose altitude is 20 in. and whose volume is 240π cu. in.

20. How many bushels are there in a conical heap of grain whose base is a circle 35 ft. in diameter, and whose height is 25 ft.?

21. A regular hexagon whose side is 6 in. revolves about one of the longer diagonals. Find the surface and the volume of solid generated.

22. Find the volumes of the right circular cones inscribed in and circumscribed about a regular tetrahedron whose edge is a m.

23. A right triangle whose legs are 15 in. and 20 in. is revolved about the hypotenuse as an axis. Find the surface and the volume of the solid generated.

In the frustum of a right circular cone:

24. If $H = 8$ in., $R = 10$ in., $r = 4$ in., find s; L; T; V.

25. If $H = 30$ cm., $s = 34$ cm., $r = 5$ cm., find R; L; T; V.

26. If $s = 19\frac{1}{2}$ ft., $R = 10\frac{1}{2}$ ft., $r = 3$ ft., find H; L; T; V.

27. How many square feet of tin are required to make a funnel 2 ft. long, if the diameters of the ends are 20 in. and 56 in., respectively?

28. A chimney 150 ft. high has a cylindrical flue 3 ft. in diameter. The bases of the chimney are circles whose diameters are 28 ft. and 7 ft. Find the number of cubic yards of masonry in the chimney.

29. A plane is passed parallel to the base of a right circular cone and $\frac{2}{5}$ the distance from the vertex to the base. Find the ratio of the smaller cone thus formed to the original cone. Compare the volume of the less cone with the frustum formed.

$$\frac{\text{Original cone}}{\text{Less cone}} = \frac{5^3}{2^3}\ (?) = \text{etc.}$$

Hence $$\frac{\text{Original cone} - \text{Less cone}}{\text{Less cone}} = \frac{125 - 8}{8}\ (?), \text{ etc.}$$

30. The altitude of a cone of revolution is 12 in. What is the altitude of the frustum of this cone that shall contain one fourth the volume of the whole cone?

31. The altitudes of two similar cylinders of revolution are 3 in. and 5 in. What is the ratio of their lateral areas? of their total areas? of their volumes?

32. The altitudes of two similar cylinders of revolution are 5 in. and 6 in., and the lateral area of the first is 200 sq. in. Find the lateral area of the second. If the volume of the first is 500 cu. in., what is the volume of the second?

33. The total areas of two similar cones of revolution are 24π sq. in. and 216π sq. in. and the radius of the first is 3 in. Find the radius of the second. The slant height of the first is 5 in. Find the lateral area of the second. Find the altitude of the first and the volume of the second.

34. The volumes of two similar cones of revolution are 27π cu. in. and 343π cu. in. The altitude of the first is 9 in. Find the altitude of the second. Find the radius of the base of each.

35. A cone of revolution whose radius is 10 in. and altitude 20 in., has the same volume as a cylinder of revolution whose radius is 15 in. Find the altitude of the cylinder.

36. A cylinder of revolution whose radius is 8 in. and altitude 30 in., is formed into a cone of revolution whose altitude is 40 in. Find the radius of its base.

37. The heights of two equivalent cylinders of revolution are in the ratio of 4 : 9. If the diameter of the first is 92 ft., what is the diameter of the second?

38. A cylinder of revolution 8 ft. in diameter is equivalent to a cone of revolution 7 ft. in diameter. If the height of the cone is 16 ft., what is the height of the cylinder?

39. Two circular cylinders having equal altitudes are to each other as their bases.

40. Two circular cylinders having equal bases are to each other as their altitudes.

41. Two circular cylinders having equal bases and equal altitudes are equal.

42. Two circular cones having equal altitudes are to each other as their bases.

43. Two circular cones having equal bases are to each other as their altitudes.

44. Two circular cones having equal bases and equal altitudes are equal.

45. If the altitude of a right circular cylinder is equal to the radius of the base, the lateral area is half the total area.

46. If the altitude of a right circular cylinder is half the radius of the base, the lateral area is equal to the area of the base.

47. If the slant height of a right circular cone is equal to the diameter of the base, the lateral area is double the area of the base.

48. The lateral area of a cone of revolution is equal to the area of a circle whose radius is a mean proportional between the slant height and the radius of the base.

49. The lateral area of a cylinder of revolution is equal to the area of a circle whose radius is a mean proportional between the altitude of the cylinder and the diameter of its base.

50. What relation does the section of a circular cone made by a plane parallel to the base have to the base? Prove.

51. At what distance from the vertex of a right circular cone whose altitude is H must a plane parallel to the base be passed, so as to bisect the lateral area? At what distance must it be passed so as to bisect the volume?

52. What does the volume V of a right circular cone become, if the altitude is doubled and the base undisturbed? Prove. What does the volume V become if the radius of the base is doubled but the altitude undisturbed? Prove. What does the volume become if both radius of base and altitude are doubled? Prove.

53. The intersection of two planes tangent to a cylinder is a line parallel to an element.

54. The intersection of two planes tangent to a cone is a line through the vertex.

55. One straight line can be drawn upon a cylindrical surface through a given point, and only one.

56. If two cylinders of revolution have equivalent lateral areas, their volumes are to each other as their radii.

57. If a rectangle is revolved about its unequal sides as axes, the volumes of the two solids generated are inversely proportional to the axes, and directly proportional to the radii of the bases.

58. Show that the formula for the volume of a circular cone can be derived from the formula for the volume of a frustum of a circular cone if one base of the frustum becomes a point.

59. Reduce the formula for the volume of a frustum of a circular cone if the radius of one base is double the radius of the other.

60. Could you prove the theorem of 680 by inscribing a frustum of a pyramid? Could you prove the theorem of 684 by circumscribing a frustum of a pyramid? Give reasons for your answer.

61. Could you prove the theorem of 678 by inscribing a pyramid? Could you prove the theorem of 682 by inscribing a pyramid? Give reasons.

62. Pass a plane tangent to a circular cylinder and containing a given element.

Construction: Draw a line in plane of base tangent to the base at the end of the given element, etc.

63. Pass a plane tangent to a circular cone and containing a given element.

64. Divide the lateral surface of a cone of revolution into two equivalent parts by a plane parallel to the base.

65. Pass a plane tangent to a circular cylinder and through a given point without it.

Construction: From the point draw a line ‖ to an element, meeting the plane of the base. From this point of intersection draw a line tangent to the base of the cylinder. Through the point of contact draw an element, etc.

66. Pass a plane tangent to a circular cone through a given point without it.

Construction: Connect this point with the vertex of the cone and prolong this line to meet the plane of the base, etc.

67. Find the locus of points at a given distance from a given straight line.

68. Find the locus of points equally distant from two given points and at a given distance from a straight line. Discuss.

69. Find the locus of points at a given distance from a given plane and at a given distance from a given line. Discuss.

70. Find the locus of points at a given distance from a given cylindrical surface whose directrix is a circle, and whose elements are perpendicular to the plane of the circle.

71. Find the locus of a point at a given distance from a given line and equally distant from two given planes. Discuss.

72. Find a point X, at a given distance from a given line and equally distant from three given points. Discuss.

73. A grain elevator in the form of a frustum of a right circular cone is 30 ft. high, and the radii of its bases are 12 ft. and 8 ft. respectively. If a bushel contains approximately $1\frac{1}{4}$ cu. ft., how many bushels of wheat will this elevator hold?

74. A certain coffee pot is 7 in. deep, $3\frac{1}{2}$ in. in diameter at the top, and 5 in. at the bottom. If there are 5 cups in a quart, how many cups of coffee will this coffee pot hold? (Disregard fractional parts of a cup.)

75. Find the radius of a circle having the same area as the lateral area of a cone of revolution whose radius is 4 and slant height is 9.

76. The frustum of a circular cone is 15 in. high and the bases are circles whose radii are 3 in. and 5 in. Find the edge of an equivalent cube.

77. The frustum of a right circular cone has a slant height of 9 ft. and the radii are 5 ft. and 7 ft. Find the lateral area and the total area. What is the length of the altitude of this frustum? Find the altitude of the cone that was removed to leave this frustum. Now find in two ways the volume of the frustum.

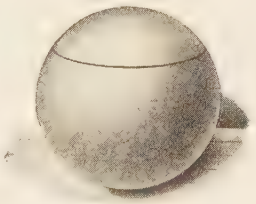
BOOK IX

THE SPHERE

687. A **sphere** is a solid bounded by a surface, all points of which are equally distant from a point within, called the **center.** The surface of a sphere is called a **spherical surface.**

A **radius** of a sphere is a straight line drawn from the center to any point of the surface.

A **diameter** of a sphere is a straight line that contains the center and has its extremities in the surface.



Ex. 1. What is the locus of all points 2 in. from the surface of a sphere whose radius is 10 in.?

Ex. 2. Consider the center of a sphere at one of the vertices of a rectangular box or room. What part of the sphere is within the box or room?

Ex. 3. Name several familiar objects that are usually regarded as spheres.

Ex. 4. What is the shape of the celestial bodies?

Ex. 5. Why do you believe the earth to be spherical?

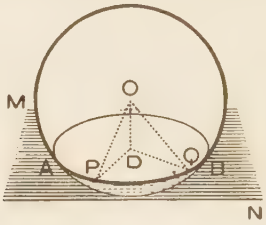
Ex. 6. Why do you believe the moon to be spherical?

Ex. 7. Describe fully the locus of points 3 in. from a line 8 in. long.

Historical Note. Archimedes was the discoverer of the formulas for the surface and volume of the sphere. Menelaus (100 A.D.) gave us the properties of spherical triangles.

Proposition I. Theorem

688. Every plane section of a sphere is a circle.



Given: Sphere whose center is O; plane MN intersecting sphere in AB.

To Prove: The figure AB is a $\odot$.

Proof: Draw $OD \perp$ to plane MN, meeting the plane at D.

Take P and Q, any two points on the perimeter of the section, and draw DP, DQ, OP, OQ.

$\triangle ODP$ and ODQ are rt. $\triangle$ (?).

In right $\triangle ODP$ and ODQ,

$OD = OD$ (Iden.).

$OP = OQ$ (687).

$\therefore$ the right $\triangle$ are congruent (?).

Hence $DP = DQ$ (?).

That is, all points of the boundary of AB are equally distant from D.

Hence the section AB is a circle (179). Q.E.D.

689. A **great circle** of a sphere is a section of the sphere made by a plane containing the center of the sphere.

A **small circle** of a sphere is a section of the sphere made by a plane that does not contain the center of the sphere.

The **axis of a circle** of a sphere is the diameter of the sphere perpendicular to the plane of the circle.

The **poles of a circle** of a sphere are the ends of its axis.

A **quadrant** (in spherical geometry) is one fourth of a great circle.

Equal spheres are spheres having equal radii.

690. A **plane** is **tangent** to a sphere if it touches the sphere in one and only one point. Two **spheres** are **tangent** to each other if they are tangent to the same plane at the same point. They may be tangent internally or externally.

A **line** is tangent to a sphere if it touches the sphere in one and only one point and does not intersect it.

A **line** is tangent to a circle of a sphere if it lies in the plane of the circle and touches the circle in one and only one point. In all cases this common point is the **point of contact** or **point of tangency.**

691. A **sphere** is **inscribed** in a polyhedron if all the faces of the polyhedron are tangent to the sphere.

A **sphere** is **circumscribed** about a polyhedron if all the vertices of the polyhedron are in the spherical surface.

692. The **distance** between two points on the surface of a sphere is the less arc of a great circle passing through them.

The distance between a point on a circle of a sphere and the nearer pole of the circle is the **polar distance** of the point.

693. The **angle between two** intersecting **curves** is the angle formed by their tangents at the point of intersection.

A **spherical angle** is the angle between two great circles of a sphere.

PRELIMINARY THEOREMS

694. THEOREM. **All radii of a sphere are equal.** (687.)

695. THEOREM. **All radii of equal spheres are equal.** (689.)

696. THEOREM. **All diameters of the same sphere or of equal spheres are equal.**

697. THEOREM. **All great circles of the same sphere or of equal spheres are equal.**

698. THEOREM. **In the same sphere or in equal spheres:**

I. Equal plane sections are equally distant from the center.

II. Plane sections equally distant from the center are equal. [Converse.]

III. Of two unequal plane sections, the greater is at the less distance from the center.

IV. Of two plane sections unequally distant from the center, the section at the greater distance is the less. [Converse.]

In each case the diameters of the sections are chords of great circles. These theorems follow from 208–211.

699. THEOREM. **Two great circles of a sphere bisect each other.** (Because they have a common diameter.)

700. THEOREM. **Every great circle of a sphere bisects the sphere and the spherical surface.** (?).

701. THEOREM. **A sphere may be generated by the revolution of a semicircle about the diameter as an axis.** (?).

702. THEOREM. **One and only one great circle can be drawn through two points, not the ends of a diameter, on the surface of a sphere.** (477.)

703. THEOREM. **One and only one circle can be drawn through three points on the surface of a sphere.** (477.)

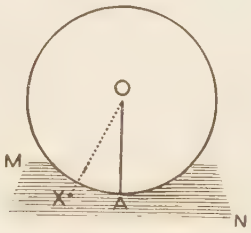
704. THEOREM. **A point is without a sphere if its distance from the center is greater than the radius, and conversely if a point is without a sphere, its distance from the center is greater than the radius.**

705. THEOREM. **The axis of a circle of a sphere passes through the center of the circle.** (Proved in proof of 688.)

THEOREMS AND DEMONSTRATIONS

Proposition II. Theorem

706. A plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere.



Given: Radius OA of sphere O, and plane $MN \perp$ to OA at A.

To Prove: MN tangent to the sphere.

Proof: Take *any* point X in MN (except A) and draw OX.

$OX > OA$ (504, I).

$\therefore$ X lies without the sphere (704).

Hence every point of plane MN, except A, is without the sphere; that is, plane MN is tangent to the sphere. (690.)

Q.E.D.

Proposition III. Theorem

707. A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact. [Converse.]

Given: Plane MN tangent to sphere O at A; radius OA.

To Prove: OA is $\perp$ to plane MN.

Proof: Every point in MN, except A, is without the sphere (690).

Take any point X in MN and draw OX.

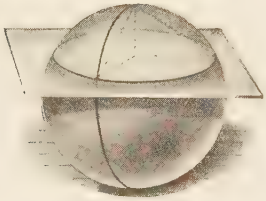
Now OX is $> OA$ (704).

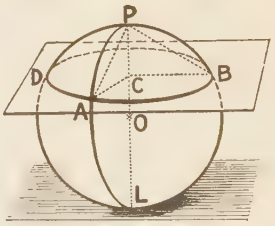
That is, OA is the shortest line from O to MN.

$\therefore$ OA is $\perp$ to MN (504, I). Q.E.D.

Proposition IV. Theorem

708. All points in the circumference of a circle of a sphere are equally distant from either pole; that is, the polar distances of all points in the circumference of a circle are equal.





Given: P and L, the poles of $\odot C$ on sphere O, and great $\textcircled{s}$ PAL, PBL.

To Prove: arc PA = arc PB,
arc AL = arc BL.

Proof: Draw the axis PL meeting plane of $\odot C$ at C. Draw AC, AP, BC, BP.

PC is $\perp$ to plane DAB (Def. of axis, 689).
$\therefore$ chord PA = chord PB (?).
Hence arc PA = arc PB (?).
Likewise arc AL = arc BL. Q.E.D.

709. Corollary. The polar distance of a great circle is a quadrant.

Ex. 1. If the radius of a sphere is 26 in. and a plane is passed, 10 in. from the center, find the radius of the circular section.

Ex. 2. What geographical circles on the earth's surface are great circles? Which are small circles?

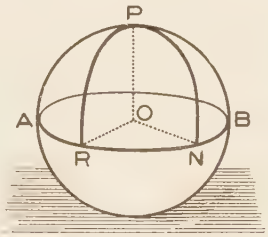
Ex. 3. Can two circles on the surface of a sphere intersect in more than two points? Why?

Ex. 4. The area of a section of a sphere 45 in. from the center is 784π sq. in. Find the radius of the sphere.

Ex. 5. The area of a section of a sphere 7 in. from the center is 576π sq. in. Find the area of a section 6 in. from the center.

Proposition V. Theorem

710. If a point on the surface of a sphere is at the distance of a quadrant from two other points on the surface, not the ends of a diameter, it is the pole of the great circle containing these two points.



Given: P, a point, and R and N, two other points (not the ends of a diameter),—all on the surface of sphere O; arcs PR and PN, quadrants; great circle $ARNB$.

To Prove: P is the pole of $\odot$ $ARNB$.

Proof: Draw the radii OP, OR, ON.

$\measuredangle$ PON and POR are rt. $\measuredangle$	(232).
$\therefore$ PO is $\perp$ to plane AB	(485).
$\therefore$ PO is the axis of $\odot$ $ARNB$	(Def. 689).
$\therefore$ P is the pole of $\odot$ $ARNB$	(Def. 689).

Q.E.D.

Ex. 1. Considering the earth as a true sphere, what is the pole of the equator? If a city has a latitude of 43°, what is its polar distance? What is the polar distance of a place upon the equator?

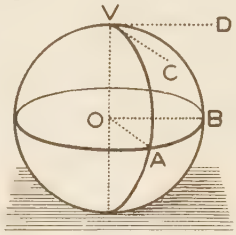
Ex. 2. In the diagram above, O is a trihedral angle. What arcs are the measures of its face angles? What is an isosceles trihedral angle? Explain by this diagram that the name is consistent with the etymology of the word.

Ex. 3. Prove that two lines tangent to a sphere at a point determine a plane tangent to the sphere at that point.

Ex. 4. Find the volume of a cube circumscribed about a sphere whose radius is 6 in. Find the volume of the cube inscribed in this sphere.

Proposition VI. Theorem

711. A spherical angle is measured by the arc of a great circle having the vertex of the angle as a pole and intersected by the sides of the angle.



Given: Spherical $\angle AVB$; arc AB of great $\odot$ whose pole is V, on sphere O.

To Prove: $\angle AVB$ is measured by arc AB.

Proof: Draw radii OA, OB, OV, and at V draw VC tangent to $\odot VA$, and VD tangent to $\odot VB$. VB is a quadrant. (709.)

OV is $\perp$ to VD (203),

OV is $\perp$ to OB (232),

OV is $\perp$ to VC and to OA (?).

$\therefore$ VD is $\parallel$ to OB, and VC is $\parallel$ to OA (62).

$\therefore \angle CVD = \angle AOB$ (499).

$\angle CVD$ is the spherical $\angle AVB$ (693).

$\angle AOB$ is measured by arc AB (232).

$\therefore \angle CVD$ is measured by arc AB (Ax. 6).

That is, $\angle AVB$ is measured by arc AB (Ax. 6).

Q.E.D.

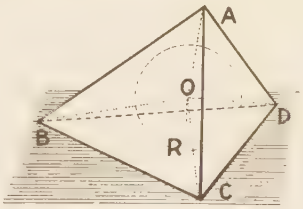
712. Corollary. **All arcs of great circles containing the pole of a great circle are perpendicular to the great circle.**

713. Corollary. **A spherical angle is equal to the plane angle of the dihedral angle formed by the planes of its sides.**

714. Corollary. **If two great circles are perpendicular to each other, each contains the pole of the other.** (528; 689.)

Proposition VII. Theorem

715. A sphere may be inscribed in any tetrahedron.



Given: Tetrahedron A–BCD. **To Prove:** (?).

Proof: Pass plane OAB bisecting dih. $\angle AB$, and plane OBC bisecting dih. $\angle BC$, and plane OCD bisecting dih. $\angle CD$, the three planes meeting at point O.

Point O, in plane OAB, is equally distant from faces ABC and ABD. (?.)

Point O, in plane OBC, is equally distant from faces ABC and BCD. (?.)

Point O, in plane OCD, is equally distant from faces BCD and ACD. (?.)

$\therefore$ O is equally distant from all four faces (Ax. 1).

Hence the sphere constructed with O as a center and OR as a radius, is tangent to each of the four faces.

$\therefore$ that sphere is inscribed in the tetrahedron (691). Q.E.D.

716. Corollary. The six planes bisecting the six dihedral angles of any tetrahedron meet in a point.

Ex. 1. The volume of any polyhedron circumscribed about a sphere, is equal to one third the product of the surface of the polyhedron and the radius of the sphere.

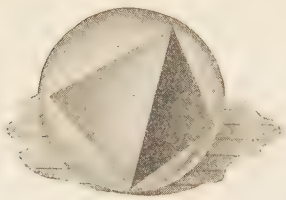
Proof: Pass planes each containing the center of the sphere and two vertices of the polyhedron. These form pyramids whose altitude . . . etc.

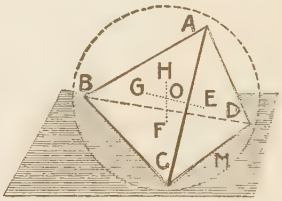
Ex. 2. What is true of the point at which the six planes meet in 716?

Ex. 3. Explain, so that a blind boy could understand it, the process of inscribing a sphere in a tetrahedron.

Proposition VIII. Theorem

717. A sphere may be circumscribed about any tetrahedron.





Given: (?). **To Prove:** (?).

Proof: Take E and F, the centers of circles circumscribed about the faces ACD and BCD, respectively. Erect EG and $FH \perp$ to these faces. Find M, the midpoint of edge CD.

EG is the locus of all points equally distant from points A, D, and C. (511.)

FH is the locus of all points equally distant from points B, C, and D. (?.)

That is, all points in EG and FH are equally distant from C and D. (Ax. 1.)

But all points equally distant from C and D are in a plane $\perp$ to CD at M. (510.)

$\therefore$ EG and FH are in this plane and are not parallel. (Not $\perp$ to the same plane.)

That is, EG and FH must intersect at O.

Hence O is equally distant from A, B, C, and D. (Ax. 1.)

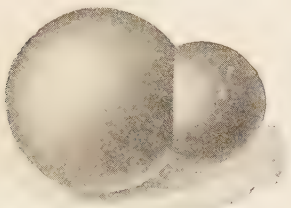
That is, using O as a center and OA, or OB, or OC, or OD, as a radius, a sphere may be circumscribed about the tetrahedron A–BCD. (691.) Q.E.D.

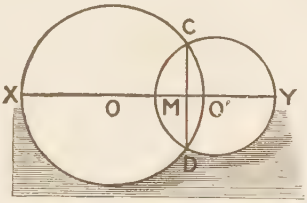
718. Corollary. Through any four points not in the same plane a sphere may be described.

719. Corollary. The six planes perpendicular to the edges of any tetrahedron at their midpoints meet in a point.

Proposition IX. Theorem

720. The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is in that line.





Given: Two intersecting circles O and O'; common chord CD; line of centers XY, intersecting CD at M.

To Prove: The spherical surfaces generated by the revolution of these ⓢ intersect in a circle.

Proof: If these ⓢ are revolved upon XY as an axis, they will generate spheres. (701.)

$CM = MD$ (219).

Point C, common to both ⓢ, will generate the intersection of the spherical surfaces. (466.)

CM is always $\perp$ to XY (219).

$\therefore$ the curved line generated by C is in one plane (486).

$\therefore$ the intersection is a circle (179).

Also the plane of this $\odot$ is $\perp$ to OO' (486).

And the center of the $\odot$ is M, in OO' (179).

Q.E.D.

Ex. 1. What can be said about the point mentioned in 719?

Ex. 2. Explain, so that a blind boy could understand, how to circumscribe a sphere about a tetrahedron.

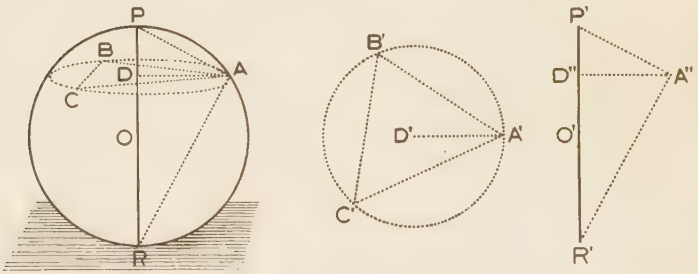
Ex. 3. Find a point equally distant from four points in space, not all in the same plane.

Ex. 4. Practical illustrations of the truth of 720 can be given by use of soap bubbles, or by two eggs.

CONSTRUCTION PROBLEMS

Proposition X. Problem

721. To find the radius of a material sphere.



Given: A material sphere. **Required**: To find its radius.

Construction: *First*, place one point of the compasses at P, and using any opening of the compasses, as AP, with the other point draw a circumference on the surface of the sphere.

Upon this circumference take three points, A and B and C, and by means of the compasses measure the straight lines AB, AC, BC.

Second, construct a $\triangle A'B'C'$, whose sides equal AB, AC, BC. Circumscribe a circle about this $\triangle$, and draw the radius $A'D'$.

Third, construct a right $\triangle P'A''D''$, whose hypotenuse equals the known line PA and whose leg equals the known radius $A'D'$. At A'' erect $A''R' \perp$ to $P'A''$ meeting $P'D''$, produced, at R'. Bisect $P'R'$ at O'.

Statement: $O'P' =$ the required radius. Q.E.F.

Proof: P is equally distant from A, B, and C (Const.).
D is equally distant from A, B, and C (179).
O is equally distant from A, B, and C (694).
$\therefore$ if the diameter $PDOR$ could be drawn it would be $\perp$ to the plane of ABC. (511.)

	$\therefore \angle PDA$ is a rt. $\angle$	(473).
Now	$\triangle ABC \cong \triangle A'B'C'$	(78).
	$\therefore DA = D'A'$	(188).
Also	$\triangle PDA \cong \triangle P'D''A''$	(84).
	$\therefore \angle P = \angle P'$	(?).
Now	$\angle PAR$ is a rt. $\angle$	(240).
Also	$\triangle PRA \cong \triangle PR'A''$	(77).
	$\therefore PR = P'R'$	(?).
	$\therefore OP = O'P'$	(Ax. 3).

That is, OP = the radius of the sphere. Q.E.D.

NOTE. The most common construction on the surface of a sphere is the drawing of a great circle. For this construction, the compasses must open so that the distance between their points is equal to the chord of a quadrant of a great circle. To find this length, as in 722, we must find the radius of the solid sphere, as in 721.

PROPOSITION XI. PROBLEM

722. To find the chord of a quadrant of a material sphere.

Given: (?).

Required: (?).

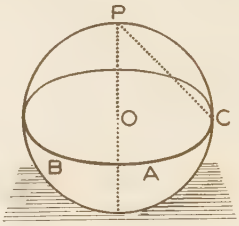
Construction: Find the radius of the sphere (by 721). Using this radius OP and any center O, describe a semicircle PMR. Erect radius $OM \perp$ to the diameter PR, and draw PM.

Statement: Arc PM is a quadrant of a great circle of the given sphere and chord PM is the required chord. Q.E.F.

Proof: Arc PM is a quadrant (?). Q.E.D.

Proposition XII. Problem

723. To describe a great circle through two given points on the surface of a sphere.



Given: The points A and B on the surface of the sphere O.

Required: To describe a great circle through A and B.

Construction: Find the chord of a quadrant of the given sphere (by 722).

Place one point of the compasses at A, and using the chord just found as an opening, describe an arc on the surface of the sphere.

Place one point of the compasses at B, and using the same opening, describe an arc, meeting the former arc at P.

Now place one point of the compasses at P and describe the circle BAC, using the same opening as before.

Statement: Circle BAC is the required great circle.

Proof: Points A and B are each at the distance of a quadrant from P. (Construction.)

$\therefore$ $\odot$ BAC is a great circle whose pole is P (710). Q.E.D.

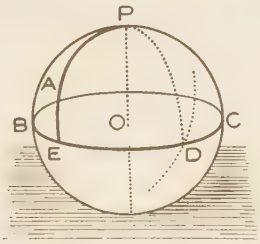
Ex. 1. A line tangent to a sphere lies in the plane tangent to the sphere at the same point.

Proof: The line is $\perp$ to the radius drawn to the point of contact (?).
The plane also is $\perp$ to the radius (?).
$\therefore$ the line is in the plane (486).

Ex. 2. At a point on the surface of a sphere there can be only one tangent plane.

Proposition XIII. Problem

724. To draw an arc of a great circle through a given point on the surface of a sphere and perpendicular to a given great circle.



Given: Point A on sphere O, and great circle BC, whose pole is P.

Required: To draw through A an arc of a great circle $\perp$ to the great circle BC.

Construction: Place one point of the compasses at A and, with an opening equal to the chord of a quadrant of the given sphere, describe an arc of a great circle intersecting the given great circle at D.

Now place one point of the compasses at D and similarly draw arc of great circle PAE. Draw PD, the arc of a great circle (by 723).

Statement:	Arc PAE is $\perp$ to circle $BEDC$.	Q.E.F.
Proof:	ED, EP, and PD are quadrants	(709).
	$\therefore$ E is the pole of arc PD	(710).
Hence	$\angle PED$ is measured by quadrant PD	(711).
	$\therefore$ $\angle PED$ is a right angle	(232).
That is,	arc PAE is $\perp$ to circle $BEDC$.	Q.E.D.

Ex. 1. Construct a plane tangent to a sphere at a given point on the surface.

Ex. 2. Construct a plane tangent to a sphere from a given point without the sphere. How many such planes are there?

SPHERICAL TRIANGLES

725. A **spherical triangle** is a portion of the surface of a sphere bounded by three arcs of great circles.

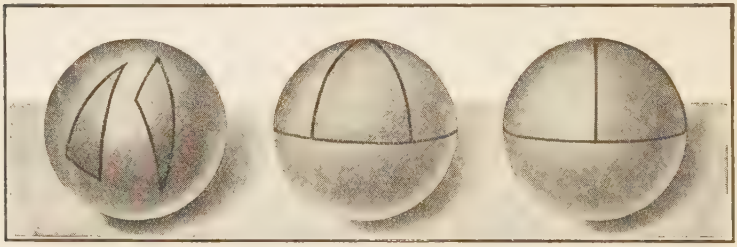
The bounding arcs are the **sides** of the triangle.

The intersections of the sides are the **vertices** of the triangle.

The angles formed by the sides are the **angles** of the triangle.

Spherical triangles are equilateral, equiangular, isosceles, scalene, acute, right, obtuse, under the same conditions as in plane triangles.

726. A **birectangular spherical triangle** is a spherical triangle two of whose angles are right angles.

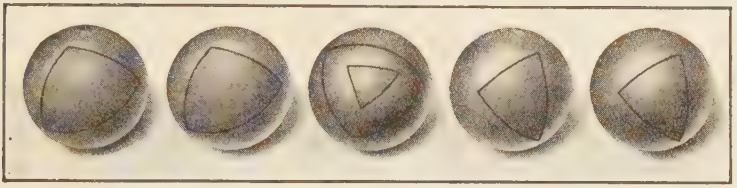


A **trirectangular spherical triangle** is a spherical triangle all of whose angles are right angles.

The **unit** usually employed in measuring the sides of a spherical triangle is the degree.

It is obvious that three great circles (not meeting at a point) divide the surface of a sphere into eight spherical triangles.

727. Two spherical triangles are **mutually equilateral** if the sides of the triangles are equal each to each; and they are **mutually equiangular** if their angles are equal each to each.



MUTUALLY EQUILATERAL SPHERICAL TRIANGLES | POLAR TRIANGLES | MUTUALLY EQUIANGULAR SPHERICAL TRIANGLES

728. If three great circles are described, having as their poles the vertices of a spherical triangle, one of the eight triangles thus formed is the **polar triangle** of the first.

The **polar triangle** is the one whose vertices are nearest the vertices of the original triangle.

729. **Symmetrical spherical triangles** are triangles that have their parts equal but arranged in reverse order. They correspond to symmetrical trihedral angles.

Vertical spherical triangles correspond to vertical trihedral angles.

If the diameters of a sphere are drawn to the vertices of a spherical triangle, the original triangle and the triangle whose vertices are the opposite ends of these diameters are **vertical spherical triangles**.

730. A **spherical polygon** is a portion of the surface of a sphere bounded by three or more arcs of great circles.

Two spherical polygons are **congruent** if they can be made to coincide. A **diagonal** of a spherical polygon is an arc of a great circle connecting two vertices not in the same side.

Only *convex* spherical polygons are considered in this book.

PRELIMINARY THEOREMS

731. Theorem. The planes of the sides of a spherical triangle form a trihedral angle:

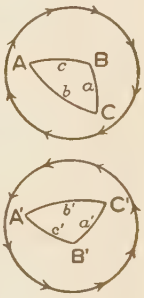
I. Whose vertex is the center of the sphere.

II. Each of whose face angles is measured by the intercepted side of the triangle. (232.)

III. Each of whose dihedral angles is equal to the corresponding angle of the triangle. (713.)

732. Theorem. Two symmetrical spherical triangles are mutually equilateral and mutually equiangular. (51, 193, 522, 713.)

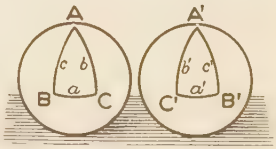
733. Theorem. The homologous parts of two symmetrical spherical triangles are arranged in reverse order.



Proof: If the eye is at the center of the sphere, the order of the vertices A, B, C is the same in direction as the motion of the hands of a clock. But the order of A', B', C' is in the opposite direction. (See 541, Note.) Hence the parts are arranged in reverse order. Q.E.D.

734. Theorem. The homologous parts of two symmetrical spherical triangles are equal. (732.)

735. Theorem. Two symmetrical isosceles spherical triangles are congruent.



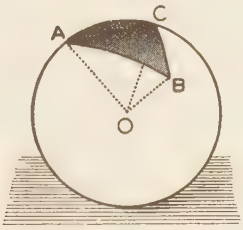
Proof: The method of superposition, as in the case of plane triangles.

Historical Note. It was not until the seventeenth century that polar triangles were invented by Gerard of Holland. It was he also who found the formulas for the area of a spherical triangle and of a spherical polygon.

THEOREMS AND DEMONSTRATIONS

PROPOSITION XIV. THEOREM

736. **One side of a spherical triangle is less than the sum of the other two.**



Given: (?). **To Prove:** $AB < AC + BC$.

Proof: Draw radii OA, OB, OC.

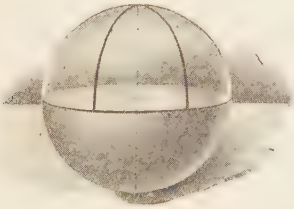
In the trihedral $\angle O$,

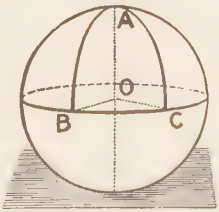
$\angle AOB < \angle AOC + \angle BOC$	(548).
$\angle AOB$ is measured by arc AB, etc.	(?).
$\therefore$ arc $AB <$ arc $AC +$ arc BC	(Ax. 6).

Q.E.D.

PROPOSITION XV. THEOREM

737. **In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the third angle is measured by the third side.**



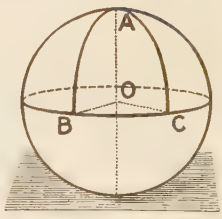


Given: Birectangular $\triangle ABC$; $\angle B$ and $\angle C$, right $\measuredangle$s.

To Prove: I. AB and AC quadrants.

II. $\angle A$ is measured by arc BC.

Proof: I. Draw radii OA, OB, OC.
Arc AB is $\perp$ to arc BC and
arc AC is $\perp$ to arc BC (Hyp.).
$\therefore$ A is the pole of arc BC (714).
$\therefore$ AB and AC are quadrants (709).
II. $\angle A$ is measured by arc BC. (711).
Q.E.D.



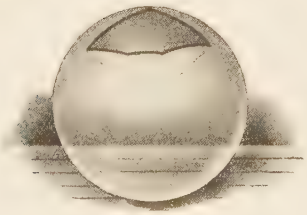
738. Corollary. The three sides of a trirectangular spherical triangle are quadrants.

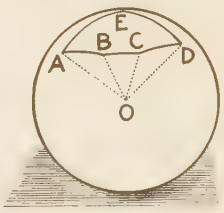
Ex. 1. If two sides of a spherical triangle are quadrants, the triangle is birectangular. (710, 712.)

Ex. 2. If all sides of a spherical triangle are quadrants, the triangle is trirectangular.

Proposition XVI. Theorem

739. The sum of the sides of any spherical polygon is less than 360°.





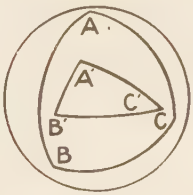
Given: (?). **To Prove**: (?).

Proof: Draw radii to the several vertices of the polygon, forming the polyhedral $\angle O$.
Then $\angle AOB + \angle BOC + \angle COD + \angle AOD < 360°$ (549).
But $\angle AOB$ is measured by arc AB, etc. (?).
$\therefore$ arcs $AB + BC + CD + AD < 360°$ (Ax. 6). Q.E.D.

740. Corollary. **The sum of the sides of any spherical polygon is less than the circumference of a great circle.**

Proposition XVII. Theorem

741. If one spherical triangle is the polar of a second triangle, then the second is the polar of the first.



Given: Spherical $\triangle ABC$ and its polar $\triangle A'B'C'$.

To Prove: $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$.

Proof:

A is the pole of arc $B'C'$ (Hyp.).

$\therefore$ B' is the distance of a quadrant from A (709).

C is the pole of arc $A'B'$ (?).

$\therefore$ B' is the distance of a quadrant from C (?).

Hence B' is the pole of arc AC (710).

Also A' is the pole of BC, and C' is the pole of AB.

$\therefore$ ABC is the polar $\triangle$ of $\triangle A'B'C'$ (728).

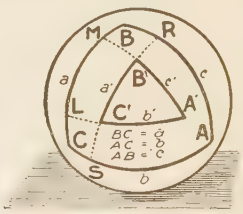
Q.E.D.

Note. Many properties of a trihedral angle are common to the corresponding spherical triangle. The polyhedral angle is similarly related to the spherical polygon. Sometimes it is advantageous to employ one, sometimes the other. The spherical triangle is perhaps simpler and more suggestive of properties than the corresponding trihedral angle, when the plane angles of its dihedral angles appear in the same diagram.

It is a most instructive and helpful exercise for the student to draw spherical triangles and their polars, etc., on a material sphere, such as a slate globe, a large apple, a ball, or other spherical object. A great many geometrical truths can be fixed in the mind by an orange and three long needles.

Proposition XVIII. Theorem

742. In two polar spherical triangles each angle of one and the opposite side of the other are supplementary.



Given: Polar $\triangle ABC$ and $A'B'C'$.

To Prove: $\angle A + a' = 180°$; $\angle A' + a = 180°$;
$\angle B + b' = 180°$; $\angle B' + b = 180°$;
$\angle C + c' = 180°$; $\angle C' + c = 180°$.

Proof: Prolong arc $B'C'$ to meet arc AB at R and arc AC at S.

$B'S = 90°$ and $C'R = 90°$ (709).

$\therefore B'S + C'R = 180°$ (Ax. 2).

That is, $C'S + B'C' + C'R$ or $RS + B'C' = 180°$ (Ax. 4).

Now RS is the measure of $\angle A$ (711).

Also $B'C' = a'$.

$\therefore \angle A + a' = 180°$ (Ax. 6).

Similarly, $\angle B + b' = 180°$; $\angle C + c' = 180°$.

Again, prolong arcs $A'B'$ and $A'C'$ to meet arc BC at M and L.

$BL = 90°$ and $CM = 90°$ (709).

$\therefore BL + CM = 180°$ (Ax. 2).

That is, $LM + MB + CM = 180°$, or $LM + BC = 180°$ (Ax. 4).

Now LM is the measure of $\angle A'$ (?).

Also $BC = a$.

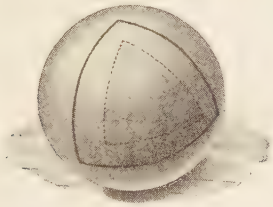
$\therefore \angle A' + a = 180°$ (Ax. 6).

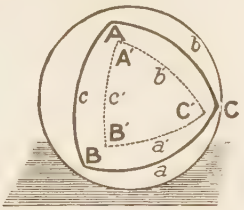
Similarly, $\angle B' + b = 180°$; $\angle C' + c = 180°$. Q.E.D.

743. COROLLARY. **In two polar spherical triangles each angle of one is measured by the supplement of the opposite side of the other.**

PROPOSITION XIX. THEOREM

744. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.





Given: A spherical $\triangle ABC$.

To Prove: I. $\angle A + \angle B + \angle C > 180°$;

II. $\angle A + \angle B + \angle C < 540°$.

Proof: I. Construct $\triangle A'B'C'$, the polar $\triangle$ of $\triangle ABC$.

$\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$ (742).

Adding, $\angle A + \angle B + \angle C + a' + b' + c' = 540°$ (Ax. 2).

But $a' + b' + c' < 360°$ (739).

Subtracting, $\angle A + \angle B + \angle C > 180°$ (Ax. 9).

Q.E.D.

II. Again $\angle A + \angle B + \angle C + a' + b' + c' = 540°$ (Ax. 2).

But $a' + b' + c' > 0°$ (725).

Subtracting, $\angle A + \angle B + \angle C < 540°$ (Ax. 9).

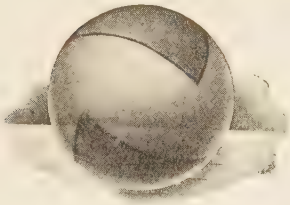
Q.E.D.

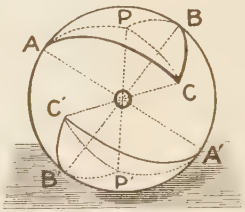
745. COROLLARY. **The sum of the angles of a spherical triangle is greater than two, and less than six, right angles.**

746. COROLLARY. **A spherical triangle may have one, two, or three obtuse angles.**

Proposition XX. Theorem

747. Two symmetrical spherical triangles are equal.





Given: Two symmetrical spherical $\triangle ABC$ and $A'B'C'$.

To Prove: $\triangle ABC = \triangle A'B'C'$.

Proof: Suppose P is the pole of the $\odot$ containing A, B, C. Draw the diameters AA', BB', CC', PP', and the arcs of great $\odot$, PA, PB, PC, $P'A'$, $P'B'$, $P'C'$.

$\angle POA = \angle P'OA'$ (?).

$\therefore$ arc PA = arc $P'A'$ (193).

Also arc PB = arc $P'B'$ and arc PC = arc $P'C'$.

But $PA = PB = PC$ (708).

$\therefore P'A' = P'B' = P'C'$ (Ax. 1).

Hence $\left.\begin{array}{l} \triangle APB \cong \triangle A'P'B' \\ \triangle ACP \cong \triangle A'C'P' \\ \triangle BPC \cong \triangle B'P'C' \end{array}\right\}$ (735).

Adding, $\triangle ABC = \triangle A'B'C'$ (Ax. 2).

Q.E.D.

Note. If the pole P should be without the $\triangle ABC$, one of the pairs of equal $\measuredangle$ would be without the original $\measuredangle$ and would be subtracted from the sum of the others to obtain $\measuredangle ABC$ and $A'B'C'$.

748. Corollary. Vertical spherical triangles are symmetrical and equal.

Ex. 1. Are symmetrical spherical triangles ever congruent?

Ex. 2. What unit is used in measuring the sides and angles of a spherical triangle?

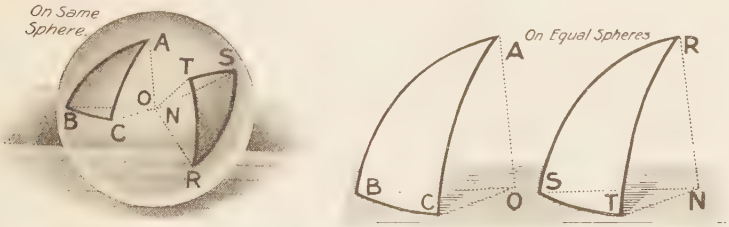
Proposition XXI. Theorem

749. Provided two spherical triangles on the same sphere (or on equal spheres) have their parts arranged in the same order, they are congruent:

I. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

II. If a side and the two angles adjoining it of one are equal respectively to a side and the two angles adjoining it in the other.

III. If three sides of the one are equal respectively to three sides of the other; that is, if they are mutually equilateral.



Given: (?).

To Prove: $\triangle ABC \cong \triangle RST$.

Proof: I and II. Superposition as in plane $\measuredangle$.

III. Draw radii of the sphere to all the vertices of the $\measuredangle$.

The face $\measuredangle$ of the trih. $\angle O$ = the face $\measuredangle$ of the trih. $\angle N$, respectively. (?.)

Hence trih. $\angle O$ = trih. $\angle N$ (546).

$\therefore$ dih. $\angle OA$ = dih. $\angle NR$; dih. $\angle OB$ = dih. $\angle NS$; etc.

$\therefore$ the $\measuredangle$ are mutually equiangular (731, III).

Hence the $\measuredangle$ can be made to coincide.

$\therefore \triangle ABC \cong \triangle RST$ (26).

Q.E.D.

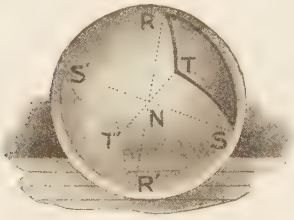
Proposition XXII. Theorem

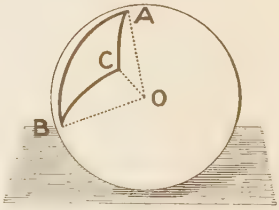
750. Provided two spherical triangles on the same sphere (or on equal spheres) have their parts arranged in reverse order, they are symmetrical:

I. If two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

II. If a side and the two angles adjoining it of one are equal respectively to a side and the two angles adjoining it of the other.

III. If three sides of one are equal respectively to three sides of the other; that is, if they are mutually equilateral.





Given: (?). **To Prove:** (?).

Proof: In each of these cases construct a third spherical $\triangle R'S'T'$, symmetrical to the $\triangle RST$.

Then $\triangle R'S'T'$ will have its parts equal to the parts of $\triangle ABC$ and arranged in the same order.

$\therefore \triangle R'S'T' \cong \triangle ABC$ (749).

Hence $\triangle RST$ is symmetrical to $\triangle ABC$ (Ax. 6).

Q.E.D.

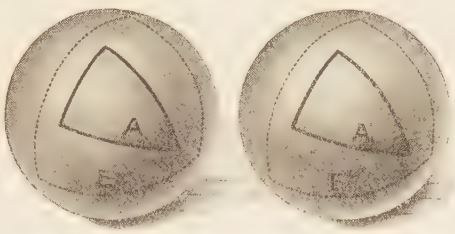
751. Corollary. **Two mutually equilateral spherical triangles are mutually equiangular and are congruent or symmetrical.**

When are they *congruent?* When are they *symmetrical?*

Ex. Is the corresponding theorem about plane triangles true?

Proposition XXIII. Theorem

752. Two mutually equiangular spherical triangles on the same sphere (or on equal spheres) are mutually equilateral, and are congruent or symmetrical.



Given: $\triangle A$ and A', mutually equiangular.

To Prove: $\triangle A$ and A' mutually equilateral, and congruent or symmetrical.

Proof: Construct $\triangle E$ and E', the polar $\triangle$ of A and A'.

The sides of E are supplements of the $\measuredangle$ of A.
The sides of E' are supplements of the $\measuredangle$ of A'. (742).

But the $\measuredangle$ of A are = respectively to the $\measuredangle$ of A' (Hyp.).

$\therefore$ $\triangle E$ and E' are mutually equilateral (49).

Hence $\triangle E$ and E' are mutually equiangular (751).

Again $\triangle A$ and A' are the polar $\triangle$ of E and E' (741).

$\therefore$ the sides of A are supplements of the $\measuredangle$ of E
the sides of A' are supplements of the $\measuredangle$ of E' (?).

Hence $\triangle A$ and A' are mutually equilateral (?).

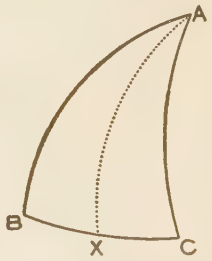
$\therefore$ they are congruent (when ?);
or symmetrical (when ?).

Q.E.D.

Ex. Is the corresponding theorem about plane triangles true? Is there any theorem concerning the congruence of plane triangles that is not true of spherical triangles? If the parts of two plane triangles were arranged in reverse order and they were kept in a plane, could they be made to coincide?

Proposition XXIV. Theorem

753. The angles opposite the equal sides of an isosceles spherical triangle are equal.



Given: (?).

To Prove: $\angle B = \angle C$.

Proof: Suppose X the midpoint of BC.

Draw AX, the arc of a great $\odot$. Now the two spherical $\triangle ABX$ and AXC are mutually equilateral. (Explain.)

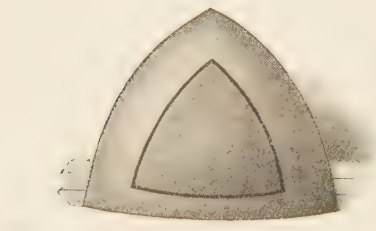
$\therefore$ they are mutually equiangular and symmetrical (751).

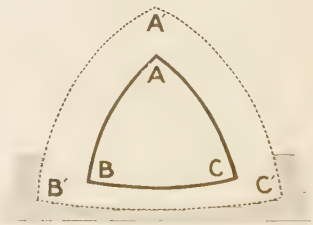
$\therefore \angle B = \angle C$ (734). Q.E.D.

754. Corollary. The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the midpoint of the base bisects the vertex angle and is perpendicular to the base.

Proposition XXV. Theorem

755. If two angles of a spherical triangle are equal, the sides opposite are equal.





Given: (?). To Prove: (?).

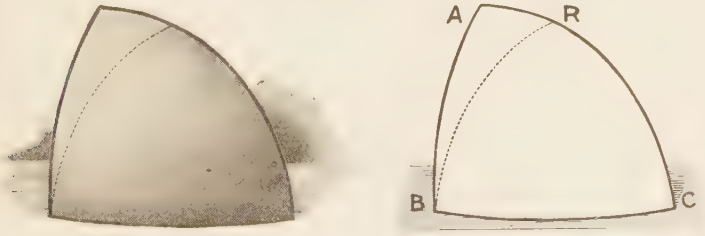
Proof: Construct $\triangle A'B'C'$, the polar $\triangle$ of $\triangle ABC$.

Then	$A'B'$ is the supplement of $\angle C$.	(?.)
And	$A'C'$ is the supplement of $\angle B$.	
	$\therefore A'B' = A'C'$	(49).
	$\therefore \angle B' = \angle C'$	(753).
Again	$\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$	(741).
	AB is the supplement of $\angle C'$, and AC of $\angle B'$	(?).
	$\therefore AB = AC$	(49).

Q.E.D.

Proposition XXVI. Theorem

756. If two angles of a spherical triangle are unequal, the sides opposite are unequal and the greater side is opposite the greater angle.



Given: $\triangle ABC$; $\angle ABC > \angle C$.

To Prove: $AC > AB$.

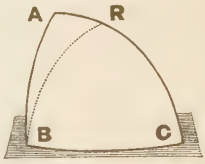
Proof: Suppose BR drawn, the arc of a great $\odot$, making $\angle CBR = \angle C$ and meeting AC at R.

Now	$AR + BR > AB$	(736).
But	$BR = CR$	(755).
	$\therefore AR + CR > AB$	(Ax. 6).
That is,	$AC > AB$.	Q.E.D.

Ex. Compare Propositions XXIV, XXV, XXVI with the corresponding theorems about plane triangles.

Proposition XXVII. Theorem

757. If two sides of a spherical triangle are unequal, the angles opposite are unequal and the greater angle is opposite the greater side. [Converse.]



Given: (?). **To Prove**: $\angle ABC > \angle C$.

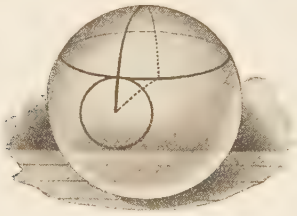
Proof: $\angle ABC$ is either $< \angle C$ or

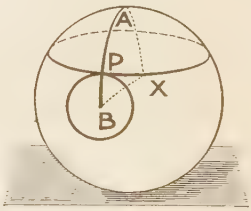
$= \angle C$ or $> \angle C$.

Continue by method of exclusion (90).

Proposition XXVIII. Theorem

758. If two circles on a sphere contain a point on the arc of a great circle that joins their poles, they have no other point in common.





Given: Point P on the arc AB of a great $\odot$ of a sphere, and P common to two circles whose poles are A and B.

To Prove: P is the only point common to these ⑤.

Proof: Suppose X is another common point.

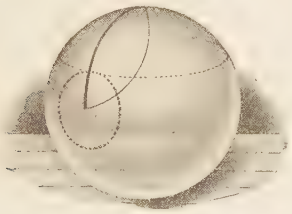
Draw arcs of great ⑤ AX and BX.

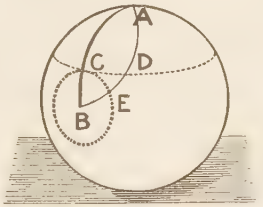
Then	$AX + BX > AP + BP$	(736).
But	$AX = AP$	(708).
Subtracting,	$BX > BP$	(Ax. 7).

That is, X is without the $\odot B$ and cannot be in both the circles. Q.E.D

Proposition XXIX. Theorem

759. The shortest line that can be drawn on the surface of a sphere, between two points on the surface, is the less arc of the great circle containing the two points.





Given: Points A and B, and AB the arc of a great ⊙ joining them; line $ADEB$, any other line on the surface of the sphere, between A and B.

To Prove: Arc $AB <$ line $ADEB$.

Proof: Take on arc AB any point C, and describe two circles through C, having A and B as their poles, and intersecting $ADEB$ at D and E. Point C is the only point common to these two ⑤. (758.)

No matter what kind of line AD is, a line of equal length can be drawn from A to C, on the surface; and a line can be drawn from B to C equal in length to BE.

[Imagine AD revolved on the surface of the sphere, using A as a pivot, and D will move along the ⊙ to point C. Similarly with BE.]

There is now a line from A to B, through C, $< ADEB$.

That is, whatever the nature of $ADEB$, there is a shorter line from A to B, which contains C, *any* point of arc AB.

Thus the shortest line contains all the points of AB and therefore *is the line* AB. Q.E.D.

Note. This theorem justifies the definition of the "distance" between two points, etc., in 692.

ORIGINAL EXERCISES

1. Vertical spherical angles are equal.

2. If two spherical triangles, on the same or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.

3. The polar triangle of an isosceles spherical triangle is isosceles.

4. The polar triangle of a birectangular spherical triangle is birectangular.

5. If two dihedral angles of a trihedral angle are equal, the opposite face angles also are equal.

Proof: Construct a sphere having the vertex as center, etc.

6. If two face angles of a trihedral angle are equal, the opposite dihedral angles also are equal.

7. A trirectangular spherical triangle is its own polar triangle.

8. Two symmetrical spherical polygons are equal.

9. Any side of a spherical polygon is less than the sum of the other sides. [Draw diagonals from a vertex.]

10. If the three face angles of a trihedral angle are equal, the three dihedral angles also are equal.

11. State and prove the converse of No. 10.

12. A straight line cannot meet a spherical surface in more than two points.

13. If two dihedral angles of a trihedral angle are unequal, the opposite face angles are unequal, and the greater face angle is opposite the greater dihedral angle.

14. State and prove the converse of No. 13.

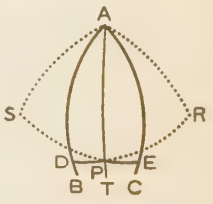
15. All the tangent lines drawn to a sphere from an external point are equal.

16. The volume of any tetrahedron is equal to one third the product of its total surface by the radius of the inscribed sphere.

17. Every point of a great circle that is perpendicular to an arc at its midpoint is equally distant from the ends of the arc.

18. The points of contact of all lines tangent to a sphere from an external point lie in the circumference of a circle.

19. Any point in the arc of a great circle that bisects a spherical angle is equally distant from the sides of the angle.



20. If the opposite sides of a spherical quadrilateral are equal, the opposite angles are equal.

21. If the opposite sides of a spherical quadrilateral are equal, the diagonals bisect each other.

22. If the diagonals of a spherical quadrilateral bisect each other, the opposite sides are equal.

23. The exterior angle of a spherical triangle is less than the sum of the opposite interior angles.

24. The sum of the angles of a spherical quadrilateral is more than four right angles.

25. If two spheres are tangent to each other, the straight line joining their centers passes through the point of contact.

26. The sum of the angles of a spherical polygon is more than $2n - 4$ right angles and less than $2n$ right angles.

27. The arcs of great circles bisecting the angles of a spherical triangle meet in a point.

28. If a tangent line and a secant are drawn to a sphere from an external point, the tangent is a mean proportional between the whole secant and the external segment.

29. The product of any secant that can be drawn to a sphere from an external point, by its external segment, is constant for all secants drawn through the same point.

30. If two spherical surfaces intersect and a plane is passed containing their intersection, tangents from any point in this plane to the two spherical surfaces are equal.

31. Find the distance from the center of a sphere whose radius is 15 in. to the plane of a small circle whose radius is 8 in.

32. The polar distance of a small circle is 60° and the radius of the sphere is 12 in. Find the radius of the circle.

33. The total surface of a tetrahedron is 90 sq. in., and the radius of the inscribed sphere is 4 in. Find the volume of the tetrahedron.

34. Find the radius of the sphere inscribed in a tetrahedron whose volume is 250 cu. in. and total surface is 150 sq. in.

35. Find the total surface of a tetrahedron whose volume is 320 cu. in., if the radius of the inscribed sphere is 8 in.

36. Find the radius of the sphere inscribed in a regular tetrahedron whose edges are each 10 in.

37. Find the radius of the sphere circumscribed about a regular tetrahedron whose edges are each 18 in.

38. Find the radii of the spheres inscribed in and circumscribed about a cube whose edges are each 10 in.

39. The sides of a spherical triangle are 60°, 80°, 110°. Find the angles of its polar triangle.

40. The angles of a spherical triangle are 74°, 119°, 87°. Find the sides of its polar triangle.

41. The chord of the polar distance of the circle of a sphere is 12 m., and the radius of the sphere is 9 m. Find the radius of the circle.

42. The polar distance of a circle is 60° and the diameter of the circle is 8 ft. Find the diameter of the sphere.

[Denote by R, each side of an equilateral triangle whose altitude is 4 ft.]

43. The radii of two spherical surfaces are 11 in. and 13 in., and their centers are 20 in. apart. Find the radius of the circle of their intersection. Find also the distances from the centers of the spheres to the center of this circle.

44. The radii of two spherical surfaces are 20 m. and 37 m., and the distance between their centers is 19 m. What is the length of the diameter of their intersection?

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45. Bisect an arc of a great circle.

46. Draw an arc of a great circle perpendicular to a given arc of a great circle through a given point in the arc.

47. Bisect a spherical angle.

48. Bisect an arc of a small circle.

49. Circumscribe a circle about a given spherical triangle.

50. Construct a spherical angle equal to a given spherical angle at a given point on the same sphere.

51. Construct a spherical triangle having the three sides given.

52. Construct a spherical triangle having the three angles given.

53. Construct a plane tangent to a sphere at a given point on the surface.

54. Construct a spherical surface having the radius given and containing three given points.

55. Construct a spherical surface that shall have a given radius, touch a given plane, and contain two given points.

56. Construct a spherical surface that shall have a given radius, shall be tangent to a given sphere, and contain two given points.

57. Construct a spherical surface that shall contain four given points.

58. Construct a plane that shall contain a given line and be tangent to a given sphere.

59. Construct a plane tangent to a given sphere and parallel to a given plane.

60. What is the locus of points on the surface of a sphere :

(*a*) Equally distant from two given points on the surface?

(*b*) Equally distant from two given points not on the surface?

61. What is the locus of the centers of those spherical surfaces that pass through two given points?

62. What is the locus of the centers of the spherical surfaces of given radius that contain two given points?

63. What is the locus of the centers of the spherical surfaces that pass through three given points?

SPHERICAL AREAS AND VOLUMES

760. A **lune** is a portion of the surface of a sphere bounded by two great semicircles.

The points of intersection of the sides of a lune are the **vertices** of the lune.

The angles made at the vertices by the sides are the **angles** of the lune.

LUNE (*a*) SPHERICAL SECTOR (*b*) SPHERICAL PYRAMID ZONE SPHERICAL CONE SPHERICAL SEGMENT

761. A **zone** is a portion of the surface of a sphere bounded by two circles whose planes are parallel.

The **bases** of a zone are the circles bounding it.

The **altitude** of a zone is the perpendicular distance between the planes of its bases.

If one of the planes is tangent to the sphere, the zone is a **zone of one base**.

762. A **spherical degree** is $\frac{1}{720}$ of the surface of a sphere. If the surface of a sphere is divided into 720 equal parts, each part is a spherical degree.

The size of a spherical degree depends on the size of the sphere.

It may be easily conceived to be half a lune whose angle is 1 degree; that is, a birectangular spherical triangle whose third angle is 1^c.

How many spherical degrees are there in a trirectangular spherical triangle?

763. The **spherical excess** of a spherical triangle is the sum of its angles less 180°. That is, $E = A + B + C - 180°$.

764. A **spherical pyramid** is a portion of a sphere bounded by a spherical polygon and the planes of its sides.

The **vertex** of a spherical pyramid is the center of the sphere.

The **base** of a spherical pyramid is the spherical polygon.

765. A **spherical sector** is the solid generated by the revolution of the sector of a circle about any diameter of the circle as an axis.

The **base** of the spherical sector is the zone generated by the arc of the circular sector.

A **spherical cone** is a spherical sector whose base is a zone of one base.

766. A **spherical segment** is a portion of a sphere included between two parallel planes that intersect the sphere.

The **bases** of a spherical segment are the circular sections made by the parallel planes.

The **altitude** of a spherical segment is the perpendicular distance between the bases.

A **spherical segment of one base** is a segment one of whose bounding planes is tangent to the sphere.

A **hemisphere** is a spherical segment of one base, which base is a great circle.

A **spherical wedge** is a portion of a sphere bounded by a lune and the planes of its sides.

Ex. 1. What is the spherical excess of a spherical triangle whose angles are 60°, 70°, and 100°?

Ex. 2. Distinguish between a zone and a spherical segment.

Ex. 3. Find the area of a spherical degree on a sphere whose surface is 3600 sq.in.

Ex. 4. Find the area of a spherical triangle containing 80 spherical degrees, on a sphere whose surface is 450 sq. ft.

Ex. 5. Find the area of a spherical polygon containing 152 spherical degrees on a sphere whose surface is 630 sq. yd.

Ex. 6. A spherical triangle containing 128 spherical degrees has an area of 72 sq. in. What is the area of the spherical surface?

PRELIMINARY THEOREMS

767. THEOREM. **Either angle of a lune is measured by the arc of a great circle described with the vertex of the lune as a pole, and included between the sides of the lune.** (711.)

768. THEOREM. **The angles of a lune are equal.**

769. THEOREM. **Every great circle of a sphere divides the sphere into two equal hemispheres, and the surface into two equal zones.**

770. THEOREM. **The spherical excess of a spherical n-gon is equal to the sum of its angles less $(n - 2)\ 180°$.**

Proof: (?).

771. Theorem. If a regular polygon having an even number of sides is inscribed in, or circumscribed about, a circle, and the figure is made to revolve about one of the longest diagonals of the polygon, the surface generated by the perimeter of the polygon approaches the surface of the sphere generated by the circle, as a limit, if the number of sides of the polygon is indefinitely increased.

772. Theorem. If a polyhedron is circumscribed about a sphere and the number of its faces is indefinitely increased, the surface of the polyhedron approaches the surface of the sphere as a limit, and the volume of the polyhedron approaches the volume of the sphere as a limit.

Note. If a regular polygon having an even number of sides is inscribed in, or circumscribed about, a circle, and the figure is made to revolve about one of the longest diagonals of the polygon, the surface generated by the polygon is composed of the surfaces of cones, a cylinder, and frustums, and the surface generated by the circle is a spherical surface.

Ex. 1. Find the spherical excess of a polygon whose angles are 80°, 110°, 140°, 130°, 160°.

Ex. 2. The spherical excess of a spherical polygon is the difference between the sum of its angles and the sum of the angles of a plane polygon having the same number of sides.

Ex. 3. The sum of the angles of a spherical quadrilateral is less than eight right angles.

Ex. 4. Find the spherical excess of a spherical hexagon if each of its angles equals 128°. If each angle equals 155°, find the excess.

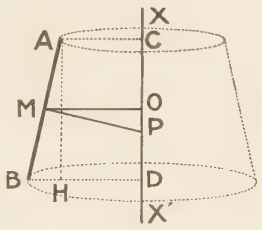
Ex. 5. If the opposite angles of a spherical quadrilateral are equal, the opposite sides are also equal.

Proof: Prolong one pair of opposite sides in both directions until they meet. Now prove two triangles congruent.

THEOREMS AND DEMONSTRATIONS

Proposition XXX. Theorem

773. **The area of the surface generated by a straight line revolving about an axis in its plane is equal to the product of the projection of the line upon the axis by the circumference of a circle whose radius is the line perpendicular to the revolving line at its midpoint, and terminating in the axis.**



Given: Line AB revolving about axis XX'; CD = projection of AB on XX'; $MP = a = \perp$ erected at midpoint of AB and terminating in XX'; MO = radius of mid-section.

To Prove: Surface generated by $AB = CD \cdot 2\pi a$.

Proof: I. The surface generated by AB is the surface of the frustum of a right circular cone whose bases are generated by AC and BD, and the mid-section, by MO.

Area of surface $= 2\pi MO \cdot AB$ (681).

Now $\triangle ABH$ and MPO are similar (310).

$\therefore MO : AH = MP : AB$ (?).

Hence $MO \cdot AB = AH \cdot MP = CD \cdot a$ (?).

$\therefore$ area of surface $= 2\pi CD \cdot a = CD \cdot 2\pi a$ (Ax. 6).

II. If AB is $\parallel$ to XX', the surface is cylindrical and equals $CD \cdot 2\pi a$ (654).

III. If AB meets XX' at C, the entire surface is conical and equals $\pi BD \cdot AB$ (680).

Now $BD = 2\,MO$ (136).

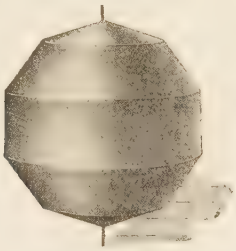
And $MO \cdot AB = CD \cdot a$. (?.)

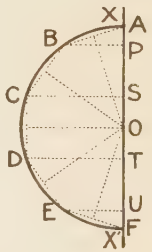
$\therefore \pi BD \cdot AB = \pi \cdot 2\,MO \cdot AB = \pi \cdot 2 \cdot CD \cdot a = CD \cdot 2\pi a$ (Ax. 6).

$\therefore$ the area of the surface $= CD \cdot 2\pi a$ (Ax. 6). Q.E.D.

Proposition XXXI. Theorem

774. The surface of a sphere is equal in area to four great circles; that is, to $4\pi R^2$.





Given: Semicircle ACF; diameter AF; S = surface of sphere generated by revolving the semicircle about AF as an axis; R = radius of this sphere.

To Prove: $S = 4\pi R^2$.

Proof: Inscribe in this semicircle half of a regular polygon having an even number of sides. Draw the apothems, a. Draw the projections of the sides of the polygon on the diameter. Now, if the figure revolves on AF as an axis,

the surface $AB = AP \cdot 2\pi a$
the surface $BC = PS \cdot 2\pi a$ (773).
the surface $CD = ST \cdot 2\pi a$ etc. Adding,

the entire surface $= (AP + PS + ST + \text{etc.}) \cdot 2\pi a$ (Ax. 2).
$= AF \cdot 2\pi a$ (Ax. 6).

Now, if the number of sides of the polygon is indefinitely increased, the entire surface generated by the polygon approaches S as a limit. (771.)

a approaches R as a limit (422).

Also $AF \cdot 2\pi a$ approaches $AF \cdot 2\pi R$.

$\therefore S = AF \cdot 2\pi R$ (229).

But $AF = 2R$ (?).

$\therefore S = 4\pi R^2$ (Ax. 6). Q.E.D.

775. COROLLARY. The area of a spherical degree equals $\frac{4\pi R^2}{720} = \frac{\pi R^2}{180}$ sq. units.

776. COROLLARY. The areas of the surfaces of two spheres are to each other as the squares of their radii and as the squares of their diameters.

Proof: $\frac{S}{S'} = \frac{4\pi R^2}{4\pi R'^2} = \frac{R^2}{R'^2} = \frac{(\frac{1}{2}D)^2}{(\frac{1}{2}D')^2} = \frac{D^2}{D'^2}$.

PROPOSITION XXXII. THEOREM

777. The area of a zone is equal to the product of its altitude by the circumference of a great circle.

Given: (The same as in 774).

To Prove: The area of the zone generated by the arc $BC = PS \times 2\pi R$.

Proof: The area generated by chord $BC = PS \cdot 2\pi a$ (773). If the number of sides of the inscribed polygon is indefinitely increased, the length of chord BC approaches arc BC and the surface generated by chord BC approaches the area of a zone.

Also $PS \cdot 2\pi a$ will approach $PS \cdot 2\pi R$.

Hence Area of zone $BC = PS \cdot 2\pi R$ (229). Q.E.D.

778. COROLLARY. Area of a zone,

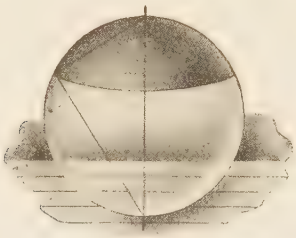
$$Z = 2\pi RH.$$

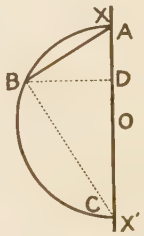
(Where Z = area of the zone, H = its altitude, and R = radius of sphere.)

Ex. 1. On a sphere whose radius is 6 in., find the area of a zone $2\frac{1}{2}$ in. in height.

Ex. 2. What does the formula for the area of a zone become when the altitude is the diameter? when the altitude is half the radius?

779. Corollary. The area of a zone of one base is equal to the area of a circle whose radius is the chord of the generating arc.





Given: Arc AB of semicircle ABC; diameter AC; chord AB.

To Prove: Area of zone generated by arc $AB = \pi \overline{AB}^2$.

Proof:

Area of zone $AB = AD \cdot 2\pi R$ (777).

That is, area of zone $AB = \pi \cdot AD \cdot 2R$.

Draw chord BC. $\triangle ABC$ is a rt. $\triangle$ (?).

$\therefore AD \cdot AC = \overline{AB}^2$ (333).

That is, $AD \cdot 2R = \overline{AB}^2$ (Ax. 6).

Hence area of zone $AB = \pi \overline{AB}^2$ (Ax. 6).

That is, **area of zone of one base $= \pi$ (chord)^2.** Q.E.D.

Ex. 1. What is the area of a zone of one base whose chord is 7 in. in length? of one whose chord is 14 in. in length?

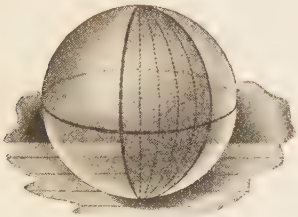
Ex. 2. What does the formula for the area of a zone of one base become when the generating arc is a semicircle? when the generating arc is a quadrant?

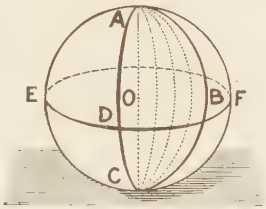
Ex. 3. If the radius of the earth is approximately 4000 mi. and the altitude of the north temperate zone is 2080 mi., what is the area of the north temperate zone?

Ex. 4. Prove that on the same or equal spheres, zones having equal altitudes have equal areas.

Proposition XXXIII. Theorem

780. The area of a lune is to the area of the surface of its sphere as the angle of the lune is to 360°.





Given: Lune $ABCDA$ on sphere O; L = area of lune; S = area of sphere; great ⊙ EB whose pole is A.

To Prove: $L : S = \angle A : 360$.

Proof: I. If arc BD and the circumference of ⊙ EB are **commensurable.** There exists a common unit of measure. Suppose this unit contained 5 times in BD; 32 times in the circumference. ∴ arc BD : circumference $= 5 : 32$ (?).

Arc BD measures $\angle A$ (711).

∴ $\angle A : 360 = 5 : 32$ (Ax. 6).

Pass great Ⓢ through the several points of division of circumference EB and vertex A, dividing the surface of the sphere into 32 equal lunes. Then $L : S = 5 : 32$ (Ax. 3).

Hence $L : S = \angle A : 360$ (Ax. 1). Q.E.D.

II. If the arc and circumference are **incommensurable.**

The proof is similar to that found in 293, 524.

781. Corollary. The number of spherical degrees in the area of a lune is double the number of degrees in its angle.

Proof: Let $L°$ denote the area of the lune, expressed in spherical degrees.

Then $L° : 720° = \angle A : 360°$ (Subst. in 780).

∴ $L° = 2 \angle A$. Q.E.D.

782. COROLLARY. **The area of a lune expressed in square units is** $\frac{\angle A \cdot \pi R^2}{90}$.

Proof: Substituting in 780, $L : 4\pi R^2 = \angle A : 360$.

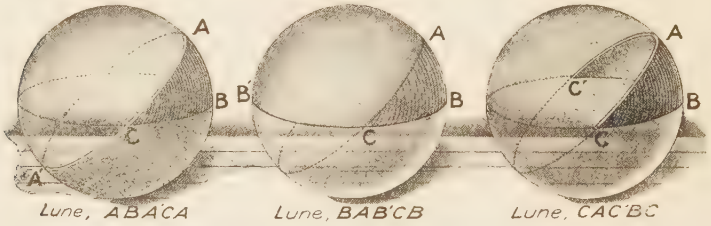
$$\therefore L = \frac{\angle A \cdot \pi R^2}{90}. \qquad \text{Q.E.D.}$$

783. COROLLARY. **Two lunes on the same or equal spheres are to each other as their angles.**

Proof: $L : S = \angle A : 360$, and $L' : S = \angle A' : 360$ (780).
Dividing, $L : L' = \angle A : \angle A'$ (Ax. 3). Q.E.D.

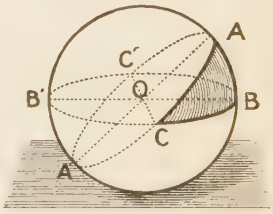
PROPOSITION XXXIV. THEOREM

784. The number of spherical degrees in a spherical triangle is equal to the spherical excess of the triangle.



Given: Spherical $\triangle ABC$ on sphere O; spherical excess of the $\triangle = E$.

To Prove: Number of spherical degrees in $\triangle ABC = E$.



Proof: Continue the sides of the $\triangle ABC$ to form the lunes $ABA'CA$, $BAB'CB$, $CAC'BC$; draw diameters AA', BB', CC'.

$$\triangle ABC' = \triangle A'B'C \qquad (747).$$

Lune $CAC'BC = \triangle ABC + \triangle AC'B = \triangle ABC + \triangle A'B'C$ (Ax. 6).

Now $\triangle ABC + \triangle A'B'C = \text{lune } CAC'BC.$
And $\triangle ABC + \triangle A'BC = \text{lune } ABA'CA.$ (Ax. 4.)
And $\triangle ABC + \triangle AB'C = \text{lune } BAB'CB.$ Adding,

$$2\triangle ABC + \triangle ABC + \triangle A'B'C + \triangle A'BC + \triangle AB'C$$
$$= \text{lune } A + \text{lune } B + \text{lune } C \quad \text{(Ax. 2).}$$

Now, *first*, 4 of these $\triangle$ compose a hemisphere and $= 360$ spherical degrees;

second, the 3 lunes $= 2\angle A + 2\angle B + 2\angle C$ (781).

By substituting in the long equation above,

$$2\triangle ABC + 360 = 2\angle A + 2\angle B + 2\angle C \quad \text{(Ax. 6).}$$

$$\therefore \triangle ABC = \angle A + \angle B + \angle C - 180 \quad \text{(Ax. 3).}$$

That is, $\triangle ABC = E$ (763). Q.E.D.

785. Corollary. The area of a spherical triangle expressed in square units is $\frac{E \cdot \pi R^2}{180}$.

Proof: 1 spherical degree $= \frac{\pi R^2}{180}$ sq. units (775).

$\therefore$ Area of a spherical $\triangle = \frac{E \cdot \pi R^2}{180}$ sq. units (Ax. 3).

Q.E.D.

Ex. 1. How many spherical degrees are there in a lune whose angle equals 25°?

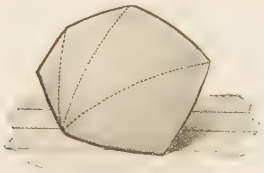
Ex. 2. On a sphere whose radius is 10 in., is a lune whose angle is 10°. Find the area of the lune in square inches.

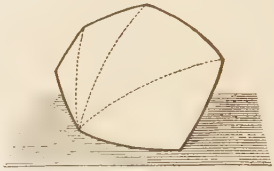
Ex. 3. Reduce the formula for the area of a lune in square units, if the angle is 90°; if the angle is 180°.

Ex. 4. On a sphere whose radius is 9 ft. is a spherical triangle whose angles are 70°, 145°, and 60°. Find the area of the triangle.

Proposition XXXV. Theorem

786. The number of spherical degrees in a spherical polygon is equal to its spherical excess.





Given: A spherical n-gon.

To Prove: The number of spherical degrees in this n-gon = the excess of the polygon.

Proof: From any vertex draw diagonals, dividing the polygon into $(n-2)$ $\triangle$; let the sums of the $\measuredangle$ of these $\triangle$ be denoted by s, s_1, s_2, $\cdots$ etc.

Now the number of spherical degrees in one $\triangle = s - 180°$ (784).

Number of spherical degrees in another $\triangle = s_1 - 180°$ (?).

Etc., for $(n-2)$ $\triangle$.

Adding, the number of spherical degrees in the n-gon = the sum of its $\measuredangle - (n-2)\,180°$ (Ax. 2).

The excess of n-gon = sum of its $\measuredangle - (n-2)\,180°$ (770).

$\therefore$ **the number of spherical degrees in a spherical polygon = the excess of the polygon** (Ax. 1). Q.E.D.

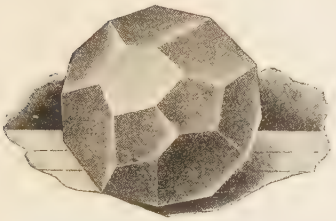
Ex. 1. Find the area of a spherical triangle whose angles are 80°, 125°, and 95°, on a sphere whose radius is 6.3 in.

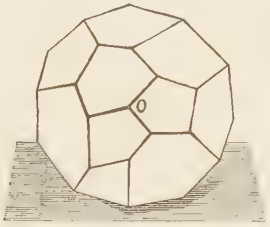
Ex. 2. Find the area of a spherical polygon whose angles are 135°, 105°, 85°, 155°, 120°, on a sphere whose radius is 15 ft.

Ex. 3. Find the area of a spherical triangle whose angles are 72°, 97°, and 101°, on a sphere whose radius is $3\frac{1}{2}$ in.

Proposition XXXVI. Theorem

787. The volume of a sphere $= \frac{4\pi R^3}{3}$.





Given: Sphere O; radius $= R$; surface $= S$; volume $= V$.

To Prove: $V = \frac{4\pi R^3}{3}$.

Proof: Suppose a polyhedron circumscribed about the sphere, its surface denoted by S', and its volume by V'.

Suppose planes are passed through the edges of the polyhedron and the center of the sphere, thus dividing the polyhedron into pyramids whose vertices are all at the center, and whose common altitude is R.

The volume of one such pyramid $= \frac{1}{3} R \cdot$ its base (612).

$\therefore$ volume of *all* pyramids $= \frac{1}{3} R \cdot$ the sum of all the bases (Ax. 2).

That is, $V' = \frac{1}{3} R \cdot S'$.

Indefinitely increase the number of faces of the polyhedron, thus indefinitely decreasing each face,

and V' approaches V as a limit
and S' approaches S as a limit } (772).

Hence $\frac{1}{3} R \cdot S'$ approaches $\frac{1}{3} R \cdot S$ as a limit.

$\therefore V = \frac{1}{3} R \cdot S$ (229).

But $S = 4\pi R^2$ (?).

$\therefore V = \frac{4\pi R^3}{3}$. (Ax. 6). Q.E.D.

788. Corollary. The volumes of two spheres are to each other as the cubes of their radii or as the cubes of their diameters.

Proof:

$$\frac{V}{V'} = \frac{4\pi R^3}{3} \div \frac{4\pi R'^3}{3} = \frac{R^3}{R'^3} = \frac{(\frac{1}{2}D)^3}{(\frac{1}{2}D')^3} = \frac{D^3}{D'^3} \qquad \text{(Ax. 6).}$$

Q.E.D.

789. Corollary. The volume of a spherical pyramid is equal to one third the product of the polygon that is its base, by the radius of the sphere.

$$V = \tfrac{1}{3}(\text{area of base})R.$$

Proof: Similar to the proof of 787.

790. Corollary. The volume of a spherical wedge is to the volume of the sphere as the angle of its base is to 360°.

Proof: Similar to the proof of 780.

791. Corollary. Volume of a spherical wedge,

$$V = \frac{\angle A \cdot \pi R^3}{270}.$$

(Where A = the $\angle$ of the lune, and R = the radius of sphere.)

Proof: $V : \frac{4}{3}\pi R^3 = \angle A : 360^\circ$ (790).

$$\therefore V = \frac{\angle A \cdot \pi R^3}{270}.$$

792. Corollary. The volume of a spherical sector is equal to one third the product of the zone that is its base by the radius of the sphere.

Proof: Similar to the proof of 787.

793. Corollary. Volume of a spherical sector or a spherical cone,

$$V = \tfrac{1}{3} Z \cdot R = \tfrac{2}{3}\pi R^2 H.$$

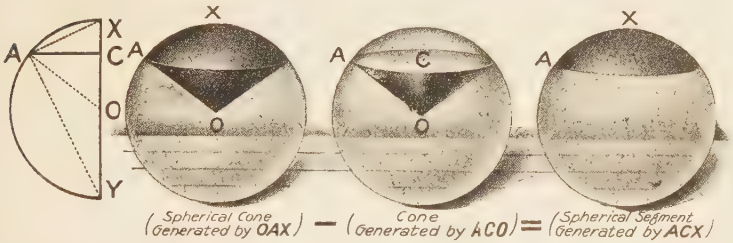
(Where V = the volume of the spherical sector or cone, H = the altitude of its base, R = the radius of the sphere.)

Proposition XXXVII. Problem

794. To derive a formula for the volume of a spherical segment.

There are Three Cases

1. Spherical segment of one base.



Given: Spherical segment generated by the figure ACX; semicircle XAY; $AC = r$; radius of sphere $= R$; altitude $= CX = H$.

Required: To find the volume of the spherical segment.

Computation: Draw chords AX, AY, and radius AO.

The right $\triangle ACO$ will generate a cone of revolution (Def. 660).

The volume of spherical segment ACX = the volume of spherical cone OAX **minus** the volume of cone ACO.

Volume of spherical cone $OAX = \frac{2}{3}\pi R^2 \cdot H$ (793).

Volume of cone $ACO = \frac{1}{3}\pi r^2 \cdot CO$ (683).

Now $r^2 = CX \cdot CY = H(2R - H)$ (331, II).

Also $CO = R - H$.

$\therefore$ vol. $ACO = \frac{1}{3}\pi H(2R - H)(R - H)$ (Ax. 6).

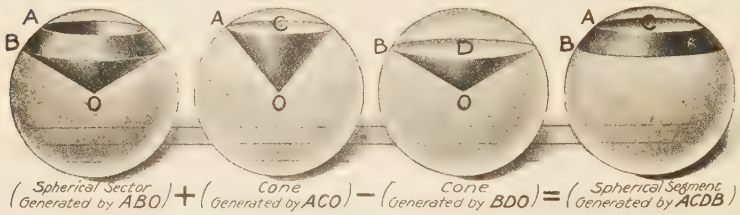
Hence volume of spherical segment

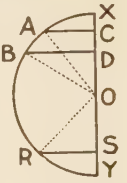
$$ACX = \tfrac{2}{3}\pi R^2 H - (\tfrac{2}{3}\pi R^2 H - \pi R H^2 + \tfrac{1}{3}\pi H^3) \quad \text{(Ax. 6).}$$

$\therefore$ **Volume of spherical segment of one base** $= \frac{1}{3}\pi H^2(3R - H)$

Q. E. F.

2. Spherical segment not including the center.





Given: Spherical segment generated by figure $ACDB$; semicircle $XABY$; $AC = r$; $BD = r'$; radius of sphere $= R$; altitude $= CD = H$.

Required: To find the volume, V, of the spherical segment.

Computation: The $\triangle ACO$ and BDO generate cones of revolution (Def. 660).

Denote OD by d.

The volume of spherical segment $ACDB$
= the volume of spherical sector ABO
plus the volume of cone ACO
minus the volume of cone BDO.

Now the volume of spherical sector $ABO = \frac{2}{3}\pi R^2 H$ (792).

And the volume of cone $ACO = \frac{1}{3}\pi r^2(d + H)$ (683).

And the volume of cone $BDO = \frac{1}{3}\pi r'^2 d$ (683).

$\therefore V = \frac{2}{3}\pi R^2 H + \frac{1}{3}\pi r^2(d + H) - \frac{1}{3}\pi r'^2 d$

$$= \frac{\pi}{3}[2R^2H + r^2H + d(r^2 - r'^2)] \quad . \quad . \quad (1)$$

But in rt. $\triangle ACO$, $R^2 = r^2 + (d + H)^2$ . . (2)
and in rt. $\triangle BDO$, $R^2 = r'^2 + d^2$ . . . . (3) } (334).

Subtracting and solving,

$$d = \frac{r'^2 - r^2 - H^2}{2H}. \quad . \quad . \quad (4)$$

Substituting in (3),

$$R^2 = \frac{r^4 + r'^4 + H^4 - 2r^2r'^2 + 2r^2H^2 + 2r'^2H^2}{4H^2} \quad (5)$$

Substituting (4) and (5) in (1), and simplifying,

$$V = \frac{\pi}{3}\left[\frac{3\,r^2H + 3\,r'^2H + H^3}{2}\right].$$

$$\therefore V = \tfrac{1}{2}\pi H(r^2 + r'^2) + \tfrac{1}{6}\pi H^3. \qquad \text{Q.E.F.}$$

3. Spherical segment including the center.

Given: Spherical segment generated by figure *BDSR*; etc.

Required: To find the volume, V, of the spherical segment.

Computation: V = the volume of spherical sector *BOR*
plus the volume of cone *BDO*
plus the volume of cone *RSO*.

(Computation similar to that in 2, with same final formula.)

ORIGINAL EXERCISES

1. Prove that the area of the surface of a sphere is equal to the square of the diameter multiplied by π; that is, $S = \pi D^2$.

2. Prove that the volume of a sphere is equal to one sixth the cube of the diameter multiplied by π; that is, $V = \frac{1}{6}\pi D^3$.

3. The surface of a sphere is equal to the cylindrical surface of the circumscribed cylinder.

4. The total surface of a hemisphere is three fourths the surface of the sphere.

5. The volume of a sphere is two thirds the volume of the circumscribed cylinder.

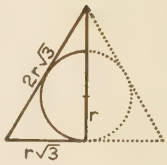
6. Upon the same circle as a base are constructed a hemisphere, a cylinder of revolution, and a cone of revolution, all having the same altitude. Prove that their total areas are $3\pi R^2$, $4\pi R^2$, $\pi R^2(1+\sqrt{2})$, respectively, and their volumes are $\frac{2}{3}\pi R^3$, πR^3, $\frac{1}{3}\pi R^3$, respectively.

7. Two zones on the same sphere, or on equal spheres, are to each other as their altitudes.

8. The area of the surface of a sphere is equal to the area of the circle whose radius is the diameter of the sphere.

9. Show that the formula for the volume of a spherical segment of one base reduces to the correct formula for the volume of a hemisphere when the base of the segment is a great circle; and to the correct formula for the volume of a sphere when the planes are both tangent.

10. In an equilateral triangle is inscribed a circle, and the figure is revolved about an altitude of the triangle as an axis. Prove:



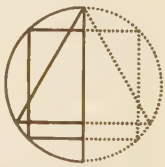
(*a*) That the surface generated by the circumference is two thirds the lateral surface generated by the triangle.

(*b*) That the volume generated by the circle is four ninths the volume generated by the triangle.

11. Derive a formula for the surface of a sphere, containing only V and π.

12. Derive a formula for the volume of a sphere, containing only S and π.

13. In a circle whose radius is R, there are inscribed a square and an equilateral triangle having their bases parallel; the whole figure is then revolved about the diameter perpendicular to the base of the triangle. Find, in terms of R:



(*a*) The total areas of the three surfaces generated.

(*b*) The volumes of the three solids generated.

14. If a cylinder of revolution having its altitude equal to the diameter of its base, and a cone of revolution having its slant height equal to the diameter of its base, are both inscribed in a sphere:

(*a*) The total area of the cylinder is a mean proportional between the area of the surface of the sphere and the total area of the cone.

(*b*) The volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

15. About a circle whose radius is a there are circumscribed a square and an equilateral triangle having their bases in the same straight line. The whole figure is then revolved about an altitude of the triangle. Find, in terms of a:

(*a*) The total areas of the three surfaces generated.

(*b*) The volumes of the three surfaces generated.

16. If a cylinder of revolution having its altitude equal to the diameter of its base, and a cone of revolution having its slant height equal to the diameter of its base, are both circumscribed about a sphere:

(*a*) The total area of the cylinder is a mean proportional between the area of the surface of the sphere and the total area of the cone.

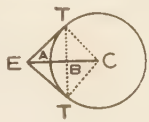
(*b*) The volume of the cylinder is a mean proportional between the volume of the sphere and the volume of the cone.

17. The line joining the centers of two intersecting spherical surfaces is perpendicular to the plane of the intersection at the center of the intersection.

18. A cube and a sphere have equal surfaces. Show that the sphere has the greater volume.

19. Prove that the parallel of latitude through a point having 30° north latitude bisects the surface of the northern hemisphere.

20. Prove that in order that the eye may observe one sixth of the surface of a sphere it must be at a distance from the center of the sphere equal to $\frac{3}{2}$ of the radius.



Proof: Zone $TT = \frac{1}{8}$ surface of sphere (Hyp.).

$\therefore AB = \frac{1}{6}$ diam. $= \frac{1}{3}R$. Hence $BC = \frac{2}{3}R$.

In rt. $\triangle ETC$, $\overline{TC}^2 = EC \cdot BC$ (?); $\therefore R^2 = EC \cdot \frac{2}{3}R$, or $EC = \frac{3}{2}R$. (Explain.) Q.E.D.

21. How many miles above the surface of the earth (diameter of earth = 7960 mi.) must a person be in order that he may see one sixth of the earth's surface?

22. If the area of a zone of one base is a mean proportional between the area of the remaining zone of the sphere and the area of the entire sphere, the altitude of the zone is $R(\sqrt{5} - 1)$.

23. The area of a lune is to the area of a trirectangular spherical triangle as the angle of the lune is to 45°.

24. A cone, a sphere, and a cylinder have the same diameters and altitudes. Prove that their volumes are in arithmetical progression.

25. The surface of a sphere bears the same ratio to the total surface of the circumscribed cylinder of revolution as the volume of the sphere bears to the volume of the cylinder.

26. The smallest circle upon a sphere whose plane passes through a given point within the sphere, is the circle whose plane is perpendicular to the diameter through the given point.

27. What part of the surface of the earth could one see if he were at the distance of a diameter above the surface?

28. Prove that if any number of lines in space are drawn through a point, and from any other point perpendiculars to these lines are drawn, the feet of all of these perpendiculars lie on the surface of a sphere.

29. The volume of a sphere is to the volume of the circumscribed cube as $\pi : 6$. The volume of a sphere is to the volume of the inscribed cube as $\pi : \frac{2}{3}\sqrt{3}$.

30. There are five spheres that touch the four planes of the faces of a tetrahedron.

31. If two angles of a spherical triangle are supplementary, the sides of the polar triangle, opposite these angles, are supplementary.

32. A square, whose side is a, is revolved about a diagonal, and also about an axis bisecting two opposite sides. Which of these figures contains the greater volume? Which has the greater surface?

33. Find the area of the surface and the volume of a sphere whose radius is 6 in.

34. Find the area of a zone whose altitude is 4 in. on a sphere whose radius is 14 in.

35. Find the area of a lune whose angle is 30° on a sphere whose radius is 8 in.

36. Find the area of a spherical triangle whose angles are 110°, 41°, 92° on a sphere whose radius is 10 in.

37. Find the volume of a sphere whose radius is 5 m.

38. Find the volume of a spherical pyramid whose base is 35 sq. in. on a sphere whose radius is 12 in.

39. Find the area of a spherical polygon whose angles are 87°, 108°, 121°, 128° on a sphere whose radius is 25 cm.

40. What is the radius of a sphere whose surface is 1386 sq. yd.?

41. What is the radius of a sphere whose volume is $\frac{500\,\pi}{3}$ cu. in.?

42. What is the area of the surface of a sphere whose volume is $288\,\pi$ cu. ft.?

43. What is the volume of a sphere the area of whose surface is 2464 sq. in.?

44. Find the area of a zone whose altitude is $3\frac{1}{2}$ in. if the radius of the sphere is $7\frac{1}{2}$ in.

45. Find the volume of a spherical sector the altitude of whose base is $5\frac{1}{4}$ in. if the radius of the sphere is 6 in.

46. Find the diameter, the circumference of a great circle, and the volume of a sphere the area of whose surface is $25\,\pi$ sq. ft.

47. By how many cubic inches is a 9-in. cube greater than a 9-in. sphere?

48. The radius of a sphere is 15 in., and the angles of the base of a spherical pyramid are 160°, 127°, 96°, 145°, and 117°. Find the volume of the pyramid.

49. A cylindrical vessel 10 in. in diameter contains a liquid. A metal ball is immersed in the liquid and the surface rises $\frac{5}{8}$ in. What is the diameter of the ball?

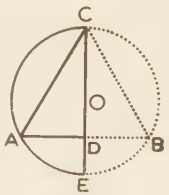
50. If a sphere 3 ft. in diameter weighs 99 lb., how much will a sphere of the same material 4 ft. in diameter weigh?

51. The radii of the bases of a frustum of a cone of revolution are 5 in. and 6 in., and the altitude of the frustum is $19\frac{1}{2}$ in. What is the diameter of an equal sphere?

52. What is the radius of a sphere whose surface is equal to the total surface of a right circular cylinder having an altitude equal to 21 in. and radius of the base equal to 6 in.?

53. Find the volume generated by the revolution of an equilateral triangle inscribed in a circle whose radius is 8 in. about an altitude of the triangle as an axis. (See Fig. of Ex. 55.)

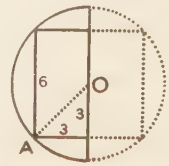
54. In the figure of Ex. 55, find the volume of the segment generated by the figure AED revolving about CD as an axis.



55. Find the area of the surface and the volume of the sphere generated by a circle that is circumscribed about an equilateral triangle whose side is 10 in.

56. Circumscribing a sphere whose radius is 18 m. is a cylinder of revolution. Compare their total areas; their volumes.

57. Circumscribing a cylinder of revolution whose altitude and diameter are each 6 in. is a sphere. Find the volume and area of the surface of the sphere.



58. Circumscribing a cylinder whose altitude is 4 in. and diameter is 3 in. is a sphere. Find the radius and volume of the sphere.

59. Each edge of a cube is 8 in. What is the area of the surface and the volume of the circumscribed sphere?

60. Find the volume of one of the segments cut from a 10 in. sphere by the plane of one of the faces of the inscribed cube.

61. The volume of a certain sphere is $179\frac{1}{2}$ cu. ft. Find the radius of a sphere 8 times as large. Find the radius of a sphere 3 times as large.

62. The radius of a certain sphere is 5 in. What is the radius of a sphere twice as great? half as great? two thirds as great?

63. A hollow sphere has an outer diameter of 20 in. and an inner diameter of 16 in. Find the volume of the metal in the shell.

64. Find the diameter of that sphere whose volume is, numerically, equal to the area of its surface.

65. A projectile consists of a right circular cylinder having a hemisphere at each end. If the cylinder is 9 in. long and 7 in. in diameter, what is the volume of one projectile?

66. Inscribed in a regular tetrahedron whose edge is 4 in., and circumscribed about it, are two spheres. Find their radii.

67. Find the radii of the spheres inscribed in and circumscribed about a regular hexahedron whose edge is 8 m.

68. Find the radii of the spheres inscribed in and circumscribed about a regular octahedron whose edge is 12 in.

69. How many spherical bullets $\frac{1}{2}$ in. in diameter can be made from a cube of lead 5 in. on each edge?

70. The area of a spherical triangle whose angles are 158°, 77°, 95° is $288\frac{3}{4}$ sq. ft. Find the radius of the sphere.

71. The area of a spherical triangle whose excess is 75° is $135\,\pi$ sq. in. Find the radius of the sphere.

72. If the radius of a sphere is 2.5 in., and the sides of a triangle on it are 104°, 115°, 101°, find the area of the polar triangle.

73. In a trihedral angle the plane angles of the dihedral angles are 75°, 85°, 110°. Find the number of degrees of surface of a sphere whose center is the vertex of the trihedral angle inclosed by the faces of this trihedral angle.

74. What is the area of a spherical hexagon, each of whose angles is 145°, on a sphere whose radius is 15 m.?

75. How many miles above the earth must a person be in order that he may see a third of its surface? one eighth of its surface?

76. Find the altitude of the zone whose area is equal to the area of a great circle of a sphere.

77. If the radius of a sphere is doubled, how is the amount of surface affected? the volume? the weight?

78. At a distance ($= d$) from the center of a sphere whose radius is r is an illuminating point. What is the altitude of the zone illuminated?

79. On a sphere having a radius of 5 in. is an equiangular spherical triangle whose area is $5\,\pi$ sq. in. Find the angles of the triangle.

80. Find the area of the surface of a sphere whose volume is 1 cu. yd.

81. Find the volume of a sphere whose surface is 1 sq. yd.

82. If a circle is described on the surface of a sphere by a pair of compasses whose points are $2\frac{1}{2}$ in. apart, what is the area of the zone bounded by this circle?

83. On a sphere the area of whose surface is 288 sq. ft. is a birectangular spherical triangle whose vertex angle is 100°. Find the area of this triangle.

84. Five inches from the center of a sphere whose diameter is two feet a plane is passed. Find the areas of the two zones formed. Find the chords of their generating arcs.

85. The diameter of the moon is about 2000 mi.; that of the earth, about 8000 mi. How do their surfaces compare? their volumes?

86. The radii of two concentric spheres are 12 in. and 13 in. A plane is tangent to the inner sphere. Find area of section of outer sphere.

87. If a solid sphere 4 ft. in diameter weighs 500 lb., what is the weight of a spherical shell, whose external diameter is 10 ft., if it is made of the same material and a foot thick?

88. The sun's diameter is about 109 times the diameter of the earth. How do the areas of their surfaces compare? their volumes?

89. How many eighth-inch spherical bullets can be made from a sphere of lead a foot in diameter?

90. A 12-inch cube of lead is melted and cast in the form of a spherical cannon ball. What is the radius of the cannon ball?

91. Find the angles of an equiangular spherical triangle equal to the sum of three equiangular spherical triangles (upon the same sphere) whose angles are each 75°.

92. What is the radius of a sphere equal to the sum of two spheres whose radii are 3 in. and 4 in. respectively?

93. What is the radius of a sphere equal to the difference of two spheres whose radii are 5 in. and 4 in. respectively?

94. The area of an equiangular spherical triangle is π sq. in., and the radius of the sphere is 4 in. Find the angles of the triangle.

95. The volumes of two spheres are to each other as 64 : 343. What is the ratio of their surfaces?

96. Find the volumes of the segments of a sphere whose radius is 12 in formed by a plane whose distance from the center is 9 in.

97. If the radius of a sphere is 20 in., find:

(a) The area of its surface.
(b) The area of a zone whose altitude is 2 in.
(c) The edge of a cube inscribed in the sphere.
(d) The area of a lune whose angle is 80°.
(e) The area of a spherical triangle whose angles are 75°, 53°, 72°.
(f) The area of a spherical polygon whose angles are 68°, 119°, 128°, 147°, 150°.
(g) The area of a birectangular spherical triangle whose vertex-angle is 54°.
(h) The area of a zone of one base whose altitude is 5 in.
(i) The radius of a sphere whose surface is four times as large.
(j) The volume of the sphere.
(k) The volume of a wedge whose angle is 36°.
(l) The volume of a spherical pyramid whose base is the triangle of exercise (e).
(m) The volume of the spherical sector whose base is the zone of exercise (b).
(n) The volume of the spherical cone whose base is the zone of exercise (h).
(o) The volume of a spherical segment of one base, whose altitude is 6 in.
(p) The volume of a spherical segment whose altitude is 4 in. and the radii of whose bases are 12 in. and 16 in.
(q) The radius of a sphere whose volume is 64 times as large.

98. The angles of a spherical triangle are 80°, 90°, 100°. Find the angle of an equal lune.

99. In a sphere whose radius is 26 in. two parallel planes are passed 34 in. apart. The radii of the two sections are 10 in. and 24 in. Find the volume of the spherical segment included between the planes.

100. In a certain refrigerating plant is a large tank of ice water. From this tank to a faucet is a pipe $\frac{3}{4}$ in. inside diameter, and 42 ft. long. The faucet is opened and 1 qt. of water runs out every 4 seconds. In what length of time will the cold water from the tank appear at the faucet?

101. A sphere 4 ft. in diameter is trisected by two concentric spherical surfaces. Find the radii of these surfaces, in inches.

102. If a cylindrical leaden bar, a ft. long and b in. in diameter, is melted and made into bullets, $\frac{1}{2}$ in. in diameter, explain the successive steps necessary to be taken to ascertain the number of bullets there will be.

SUMMARY OF FORMULAS OF SOLID GEOMETRY

B = area of base.
b = area of upper base.
E = number of edges.
e, e' = homologous edges.
F = number of faces.
H = altitude.
L = lateral area.
M = mid-section.
m = radius of mid-section.
P = perimeter of base.
P_r = perimeter of right section.
p = perimeter of upper base.
R, r = radius of base.
s = slant height.
T = total area.
V = volume; number of vertices.

Prisms and Pyramids

Parallelepiped	$V = BH$	(584).
Prism	$L = \textit{lat. edge} \cdot P_r$	(569).
	$T = L + 2B$	(570).
	$V = B \cdot H$	(590).
Prismatoid	$V = \frac{1}{6}H(b + B + 4M)$	(628).
Regular Pyramid	$L = \frac{1}{2}P \cdot s$	(604).
	$T = L + B$	(595).
Pyramid	$V = \frac{1}{3}B \cdot H$	(612).
Frustum of regular pyramid	$L = \frac{1}{2}(P + p)s$	(605).
Frustum of any pyramid	$V = \frac{1}{3}H(B + b + \sqrt{B \cdot b})$	(618).
Polyhedron	$E + 2 = V + F$	(624).
	Sum of face $\measuredangle$s $= (V - 2)360°$	(626).
Similar polyhedrons	$T : T' = e^2 : e'^2$	(632).
	$V : V' = e^3 : e'^3$	(634).

Cylinders and Cones

Right circular cylinder	$L = 2\pi RH$	(654).
	$T = 2\pi R(H + R)$	(654).
Circular cylinder	$V = \pi R^2 H$	(656).
Right circular cone	$L = \pi Rs$	(679)
	$T = \pi R(s + R)$	(679).
Circular cone	$V = \frac{1}{3}\pi R^2 H$	(683).
Frustum of right circular cone	$L = \pi(R + r)s$	(681).
	$= 2\pi ms$	(681).
	$T = \pi[(R + r)s + R^2 + r^2]$	(681)
	$V = \frac{1}{3}\pi H[R^2 + r^2 + R \cdot r]$	(685)

A = angle of lune.
E = spherical excess.
H = altitude.
L = area of lune.
R = radius of sphere.
r, r' = radii of bases of spherical segment.
S = area of spherical surface.
V = volume.
Z = area of zone.
Δ = area of triangle.

The Sphere

Spherical surface	$S = 4\pi R^2$	(774).
Zone	$Z = 2\pi RH$	(778).
Zone of one base	$Z = \pi\,(\text{chord})^2$	(779).
Lune	$L = 2A$ spherical degrees	(781).
	or $L = \frac{\angle A \cdot \pi R^2}{90}$ square units	(782).
Spherical triangle	$\Delta = E$ spherical degrees	(784).
	or $\Delta = \frac{E \cdot \pi R^2}{180}$ square units	(785).
Spherical polygon	Polygon $= E$ spherical degrees	(786).
Sphere	$V = \frac{4\pi R^3}{3}$	(787).
Spherical pyramid	$V = \frac{1}{3}\,(\text{base})\,R$	(789).
Spherical wedge	$V = \frac{\angle A \cdot \pi R^3}{270}$	(791).
Spherical sector	$V = \frac{2}{3}\pi R^2 H$	(793).
Spherical cone	$V = \frac{2}{3}\pi R^2 H$	(795).
Spherical segment, one base,	$V = \frac{1}{3}\pi H^2(3R - H)$	(794).
Any spherical segment	$V = \frac{1}{2}\pi H(r^2 + r'^2) + \frac{1}{6}\pi H^3$	(794).

INDEX OF DEFINITIONS

(The numbers refer to pages.)

Abbreviations, 6.
Acute angle, 3.
Acute triangle, 5.
Adjacent angles, 2.
Adjacent dihedral angles, 290.
Alternate angles, 22.
Alternate-exterior angles, 22.
Alternate-interior angles, 22.
Alternate-interior dihedral angles, 291.
Alternation, 145.
Altitude, of cone, 377.
  of cylinder, 367.
  of frustum, 333, 378.
  of parallelogram, 47.
  of prism, 314.
  of prismatoid, 352.
  of pyramid, 332.
  of spherical segment, 429.
  of trapezoid, 47.
  of triangle, 37.
  of zone, 428.
Analysis, 131.
Angle, acute, 3.
  between intersecting curves, 395.
  birectangular trihedral, 307.
  bisector of, 4, 37.
  central, 76.
  central, of regular polygon, 229.
  complement of, 3.
  convex polyhedral, 305.
  defined, 2.
  degree of, 3.
  dihedral, 290.
  exterior, of polygon, 60.
  exterior, of triangle, 39.
  included, 4.
  inscribed in circle, 76.
  inscribed in segment, 101.
  isosceles trihedral, 307.
  measure of, 101.
  oblique, 3.
  obtuse, 3.
Angle, plane, 2.
  plane angle of dihedral, 291.
  polyhedral, 305.
  rectangular trihedral, 307.
  right, 2.
  sides of, 2.
  spherical, 395.
  straight, 3.
  supplement of, 3.
  trihedral, 307.
  trirectangular trihedral, 307.
  vertex of, 2.
Angles, adjacent, 2.
  adjacent dihedral, 290.
  alternate, 22.
  alternate-exterior, 22.
  alternate-interior, 22.
  complementary, 3.
  complementary dihedral, 291.
  consecutive, 47.
  corresponding, 22.
  equal, 5.
  equal dihedral, 291.
  equal polyhedral, 306.
  face, of polyhedral, 305.
  homologous, of polygons, 61.
  interior, 22.
  of lune, 427.
  of polygon, 60.
  of quadrilateral, 47.
  of spherical triangle, 408.
  of triangle, 4.
  opposite, of parallelogram, 47.
  opposite interior, 39.
  right dihedral, 291.
  supplementary, 3.
  supplementary dihedral, 291.
  symmetrical polyhedral, 306.
  vertical, 2.
  vertical dihedral, 291.
  vertical polyhedral, 306.
Antecedents, 143.

Apothem, 229.
Arc, defined, 6, 76.
  degree of, 101.
  intercepted, 76.
  subtended, 76.
Arcs, similar, 237.
Area, 193.
Auxiliary lines, 19.
Axiom, 6.
Axioms, 6.
Axis, of circle of sphere, 394.
  of circular cone, 377.
  of symmetry, 65.

Base, of cone, 377.
  of figure, 47.
  of pyramid, 332.
  of spherical pyramid, 428.
  of spherical sector, 428.
  of triangle, 5.
Bases, of cylinder, 367.
  of parallelogram, 47.
  of prism, 314.
  of spherical segment, 429.
  of trapezoid, 47.
  of zone, 428.
Birectangular spherical triangle, 408.
Birectangular trihedral angle, 307.
Bisector of angle, 4, 37.
Boundary, 1.

Center, figure symmetrical with respect to, 65.
  of circle, 6, 75.
  of regular polygon, 229.
  of sphere, 393.
  of symmetry, 65.
Central angle, defined, 76.
  of regular polygon, 229.
Chord, 75.
Circle, angle inscribed in, 76.
  center of, 6, 75.
  circumscribed about polygon, 94.
  defined, 6, 75.
  diameter of, 6, 75.
  inscribed in polygon, 94.
  radius of, 6, 75.
  sector of, 76.
  segment of, 76.
  tangent to, 75.
Circles, concentric, 76.
  equal, 76.
  escribed, 125.
Circles, line tangent to, 395.
  of sphere, 394.
  tangent externally, 76.
  tangent internally, 76.
Circular cone, axis of, 377.
  defined, 377.
  right, 377.
Circular cylinder, defined, 368.
  right, 368.
Circumference, 6, 75.
Circumscribed circle, 94.
Circumscribed frustum of pyramid, 378.
Circumscribed polygon, 94.
Circumscribed prism, 368.
Circumscribed sphere about polyhedron, 395.
Commensurable quantities, 96.
Common tangent, 75.
Complement of angle, 3.
Complementary angles, 3.
Complementary dihedral angles, 291.
Composition, 145.
Composition and division, 146.
Concave polygon, 61.
Concentric circles, 76.
Conclusion, 20.
Cone, altitude of, 377.
  base of, 377.
  circular, 377.
  defined, 377.
  frustum of, 378.
  lateral area of, 377.
  lateral area of frustum of, 378.
  oblique circular, 377.
  of revolution, 378.
  plane tangent to, 378.
  right circular, 377.
  spherical, 428.
  total area of, 377.
  total area of frustum of, 378.
Cones, similar, of revolution, 378.
Congruent figures, 5.
Congruent polygons, 61.
Congruent solids, 315.
Congruent spherical polygons, 409.
Conical surface, 377.
Consecutive angles, 47.
Consequents, 143.
Constant, 96.
Construction, 117.
Continued proportion, 143.
Converse of theorem, 20.
Convex polygon, 61.

Convex polyhedral angle, 305.
Convex polyhedron, 313.
Corollary, 8.
Corresponding angles, 22.
Corresponding dihedral angles, 291.
Cube, 315, 354.
Curved line, 6, 75.
Cylinder, altitude of, 367.
bases of, 367.
circular, 368.
defined, 367
lateral area of, 367.
oblique, 368.
of revolution, 368.
right, 368.
right circular, 368.
right section of, 368.
total area of, 367.
Cylinders, similar, of revolution, 368.
Cylindrical surface, 367.

Decagon, 62.
Degree, of angle, 3.
of arc, 101.
spherical, 428.
Demonstration, defined, 8.
elements of, 20.
Determination, of a circle, 117.
of a straight line, 1.
Determined, plane, 262.
Diagonal, of polyhedron, 313.
of rectilinear figure, 47.
of spherical polygon, 409.
Diameter, of circle, 6, 75.
of sphere, 393.
Dihedral angle, defined, 290.
edge of, 290.
faces of, 290.
plane angle of, 291.
Dihedral angles, adjacent, 290.
alternate-interior, 291.
complementary, 291.
corresponding, 291.
equal, 291.
right, 291.
supplementary, 291.
vertical, 291.
Dimensions of parallelepiped, 315.
Direct proportion, 166.
Directrix, 367, 377.
Discussion, 117.
Distance between points of surface of sphere, 395.
Distance, between two points, 29.
from a point to a line, 36.
from a point to a plane, 286.
Division, 146.
Dodecagon, 62.
Dodecahedron, defined, 313.
regular, 354.

Edge, of dihedral angle, 290.
Edges, of polyhedral angle, 305.
of polyhedron, 313.
Element, of conical surface, 377.
of cylindrical surface, 367.
Equal angles, 5.
Equal circles, 76.
Equal dihedral angles, 291.
Equal figures, 5.
Equal polyhedral angles, 306.
Equal solids, 315.
Equal spheres, 395.
Equiangular polygon, 60.
Equiangular spherical triangles, 408.
Equiangular triangle, 5.
Equilateral polygon, 60.
Equilateral spherical triangles, 408.
Equilateral triangle, 5.
Escribed circles, 125.
Excess, spherical, 428.
Exterior angle, of polygon, 60.
of triangle, 39.
Extreme and mean ratio, 188.
Extremes, 143.

Face angles of polyhedral angle, 305.
Faces, of dihedral angle, 290.
of polyhedral angle, 305.
of polyhedron, 313.
Figure, base of, 47.
rectilinear, 1.
symmetrical, 65.
Figures, congruent, 5.
equal, 5.
isoperimetric, 254.
Foot, of line, 261.
of perpendicular, 2.
Fourth proportional, 143.
Frustum of cone, altitude of, 378.
defined, 378.
lateral area of, 378.
mid-section of, 378.
slant height of, 378.
total area of, 378.
Frustum of pyramid, altitude of, 333.

Frustum of pyramid, circumscribed about frustum of cone, 378.
inscribed in frustum of cone, 378.
slant height of, 333.

Generatrix, 367, 377.
Geometrical solid, 316.
Geometry, defined, 1.
plane, 1.
solid, 261.
Great circle of sphere, 394.
axis of, 394.

Harmonic division, 154.
Hemisphere, 429.
Heptagon, 62.
Hexagon, 62.
Hexahedron, defined, 313.
regular, 354.
Historical notes:
Archimedes, 77, 393.
August, 353.
Descartes, 258.
Earliest geometrical knowledge, 8.
Euclid, 17, 45.
Eudoxus, 149, 345.
Gerard, 410.
Hippasus, 355.
Hippocrates, 126.
Leibnitz, 235.
Lincoln, 217.
Menelaus, 393.
Newton, 201.
Plato, 131.
Pythagoras, 85, 355.
Thales, 103, 160.
Homologous angles, in polygons, 61.
Homologous parts, 5.
Homologous sides, in polygons, 61.
in triangles, 161.
Hypotenuse, 5.
Hypothesis, 19.

Icosahedron, defined, 313.
regular, 354.
Inclination of line, 286.
Included angles, 4.
Incommensurable quantities, 96.
Indirect method, 35.
Inscribed angle, in circle, 76.
in segment, 101.
Inscribed circle, 94.
Inscribed frustum of pyramid, 378.
Inscribed polygon, 94.
Inscribed prism, 338.
Inscribed prism in cylinder, 368.
Inscribed sphere in polyhedron, 395.
Intercept, to, 56, 76.
Intercepted arc, 76.
Interior angles, 22.
Intersect, to, 56.
Intersection, 261.
Inverse proportion, 166.
Inversion, 145.
Isoperimetric figures, 254.
Isosceles trapezoid, 47.
Isosceles triangle, 5.
Isosceles trihedral angle, 307.

Lateral area, of cone, 377.
of cylinder, 367.
of frustum of cone, 378.
of prism, 314.
of pyramid, 332.
Lateral edges, of prism, 314.
of pyramid, 332.
Lateral faces, of prism, 314.
of pyramid, 332.
Legs, of isosceles trapezoid, 47.
of right triangle, 5.
Limit of variable, 96, 97.
Limits, theorem of, 98.
Line, curved, 6.
defined, 1.
divided harmonically, 154.
divided into extreme and mean ratio, 188.
foot of, 261.
inclination of, 286.
projection of, 262.
straight, 1.
straight, oblique to plane, 261.
straight, parallel to plane, 261.
straight, perpendicular to plane 261.
tangent to circle, 75.
tangent to circle of sphere, 395.
tangent to sphere, 395.
Lines, auxiliary, 19.
divided proportionally, 149.
oblique, 2.
parallel, 3.
perpendicular, 2.
Locus, 114.
Lune, angles of, 427.
defined, 427.
vertices of, 427.

Maximum, 254.
Mean proportional, 143.
Means, 143.
Measure, of angle, 101.
  of quantity, 96.
  unit of, 96.
Median, of trapezoid, 47.
  of triangle, 37.
Method, indirect, 35.
  of exclusion, 35.
Mid-section, of frustum of cone, 378.
  of prismatoid, 352.
Minimum, 254.
Motion, 2.
Mutually equiangular polygons, 61.
Mutually equiangular spherical triangles, 408.
Mutually equilateral polygons, 61.
Mutually equilateral spherical triangles, 408.

N-gon, 62.
Normal, 261.

Oblique angle, 3.
Oblique circular cone, 377.
Oblique cylinder, 368.
Oblique lines, 2.
Oblique parallelepiped, 315.
Oblique prism, 314.
Obtuse angle, 3.
Obtuse triangle, 5.
Octagon, 62.
Octahedron, defined, 313.
  regular, 354.
Opposite angles of parallelogram, 47.
Opposite interior angles, 39.

Parallel lines, 3.
Parallel planes, 261.
Parallelepiped, 315.
  dimensions of, 315.
  oblique, 315.
  rectangular, 315.
  right, 315.
Parallelogram, altitude of, 47.
  bases of, 47.
  defined, 47.
Parts, homologous, 5.
Pentadecagon, 62.
Pentagon, 62.
Perimeter, 94.
Perpendicular, defined, 2.
Perpendicular, foot of, 2.
Perpendicular planes, 291.
Pi (π), 235.
Plane, defined, 1, 261.
  determined, 262.
  distance from point to, 286.
  line oblique to, 261.
  line parallel to, 261.
  line perpendicular to, 261.
  tangent to cone, 378.
  tangent to cylinder, 368.
  tangent to sphere, 395.
Plane angle, 2.
Plane angle of dihedral angle, 291.
Plane geometry, 1.
Plane section of polyhedral angle, 305
Planes, parallel, 261.
  perpendicular, 291.
Point, defined, 1.
  equally distant from two lines, 36.
  of contact, 75, 395.
  of tangency, 75, 395.
  projection of, 170, 262.
Points, symmetrical, 65.
Polar distance, 395.
Polar triangle, 409.
Poles of circle of sphere, 394.
Polygon, angles of, 60.
  apothem of regular, 229.
  center of regular, 229.
  central angle of regular, 229.
  circumscribed, 94.
  concave, 61.
  convex, 61.
  defined, 60.
  equiangular, 60.
  equilateral, 60.
  exterior angle of, 60.
  inscribed, 94.
  radius of, 229.
  reëntrant, 61.
  regular, 225.
  spherical, 409.
  vertices of, 60.
Polygons, congruent, 61.
  mutually equiangular, 61.
  mutually equilateral, 61.
  similar, 154.
Polyhedral angle, convex, 305.
  defined, 305.
  edges of, 305.
  face angles of, 305.
  faces of, 305.

Polyhedral angle, plane section of, 305.
vertex of, 305.
Polyhedral angles, equal, 306.
symmetrical, 306.
vertical, 306.
Polyhedron, convex, 313.
defined, 313.
diagonal of, 313.
edges of, 313.
faces of, 313.
regular, 353.
sphere inscribed in, 395.
vertices of, 313.
Polyhedrons, similar, 353.
Pons asinorum, 17.
Postulate, 7.
Prism, altitude of, 314.
bases of, 314.
circumscribed about cylinder, 368.
circumscribed about pyramid, 338.
defined, 314.
inscribed in cylinder, 368.
inscribed in pyramid, 338.
lateral area of, 314.
lateral edges of, 314.
lateral faces of, 314.
oblique, 314.
regular, 314.
right, 314.
right section of, 315.
total area of, 314.
triangular, 314.
truncated, 314.
Prismatoid, altitude of, 352.
defined, 351.
mid-section of, 352.
Prismoid, 352.
Problem, 117.
Projection, of line, 170, 262.
of point, 170, 262.
Proof, 8.
Proportion, antecedents of, 143.
consequents of, 143.
continued, 143.
defined, 143.
direct, 166.
extremes of, 143.
inverse, 166.
means of, 143.
reciprocal, 166.
Proportional, fourth, 143.
mean, 143.
third, 143.
Proposition, 8, 117.
Pyramid, altitude of, 332.
altitude of frustum of, 333.
base of, 332.
circumscribed about cone, 378.
circumscribed frustum of, 378.
defined, 332.
frustum of, 333.
inscribed frustum of, 378.
inscribed in cone, 378.
lateral area of, 332.
lateral edges of, 332.
lateral faces of, 332.
regular, 332.
slant height of frustum of regular, 333.
slant height of regular, 333.
spherical, 428.
total area of, 332.
triangular, 332.
truncated, 333.
vertex of, 332.
Pyramids, 332.

Q.E.D., 12.
Q.E.F., 117.
Quadrant, 76, 395.
Quadrilateral, angles of, 47.
defined, 47, 62.
sides of, 47.
vertices of, 47.
Quantities, commensurable, 96.
incommensurable, 96.

Radius, of circle, 6, 75.
of regular polygon, 229.
of sphere, 393.
Ratio, defined, 96, 143.
extreme and mean, 188.
series of equal, 143.
Reciprocal proportion, 166.
Rectangle, 47.
Rectangular parallelepiped, 315.
Rectangular trihedral angle, 307.
Rectilinear figure, 1.
Reductio ad absurdum, 35, 126.
Reëntrant polygon, 61.
Regular dodecahedron, 354.
Regular hexahedron, 354.
Regular icosahedron, 354.
Regular octahedron, 354.
Regular polygon, 225.
Regular polyhedron, 353, 354.

Regular prism, 314.
Regular pyramid, 332.
slant height of, 333.
Regular tetrahedron, 354.
Revolution, cone of, 378.
cylinder of, 368.
similar cones of, 378.
similar cylinders of, 368.
Rhomboid, 47.
Rhombus, 47.
Right angle, 2.
Right circular cone, 377.
Right circular cylinder, 368.
Right cylinder, 368.
Right dihedral angles, 291.
Right parallelepiped, 315.
Right prism, 314.
Right section, of cylinder, 368.
of prism, 315.
Right triangle, 5.

Scalene triangle, 5.
Secant, 75.
Sector, base of spherical, 428.
of circle, 76.
spherical, 428.
Sectors, similar, 237.
Segment, base of spherical, 429.
of circle, 76.
spherical, 428.
Segments, of line, 149.
similar, 237.
Semicircle, 76.
Series of equal ratios, 143.
Sides, homologous, in polygons, 61.
homologous, in triangles, 161.
of angle, 2.
of polygon, 61.
of quadrilateral, 47.
of spherical triangle, 408.
of triangle, 4.
Similar arcs, 237.
Similar cones of revolution, 378.
Similar cylinders of revolution, 368.
Similar polygons, 154.
Similar polyhedrons, 353.
Similar sectors, 237.
Similar segments, 237.
Similar triangles, 154.
Slant height, of cone of revolution, 378.
of frustum of cone, 378.
of frustum of pyramid, 333.
of regular pyramid, 333.
Small circle of sphere, 394.
Solid, defined, 1, 261.
geometrical, 316.
volume of, 315.
Solid geometry, 261.
Solids, congruent, 315.
equal, 315.
Sphere, axis of circle of, 394.
center of, 393.
circumscribed about polyhedron 395.
defined, 393.
diameter of, 393.
great circle of, 394.
inscribed in polyhedron, 395.
line tangent to, 395.
line tangent to circle of, 395.
plane tangent to, 395.
radius of, 393.
small circle of, 394.
Spheres, equal, 395.
tangent, 395.
Spherical angle, 395.
Spherical cone, 428.
Spherical degree, 428.
Spherical excess, 428.
Spherical polygon, defined, 409.
diagonal of, 409.
Spherical polygons, congruent, 409.
Spherical pyramid, base of, 428.
defined, 428.
vertex of, 428.
Spherical sector, base of, 428.
defined, 428.
Spherical segment, altitude of, 429.
bases of, 429.
defined, 429.
of one base, 429.
Spherical surface, 393.
Spherical triangle, angles of, 408.
birectangular, 408.
defined, 408.
sides of, 408.
symmetrical, 409.
trirectangular, 408.
unit of measure of, 408.
vertices of, 408.
Spherical triangles, mutually equiangular, 408.
mutually equilateral, 408.
Spherical wedge, 429.
Square, 47.
Statement, 117.

Straight angle, 3.
Straight line, defined, 1.
  divided harmonically, 154.
  divided, into extreme and mean ratio, 188.
  oblique to plane, 261.
  parallel to plane, 261.
  perpendicular to plane, 261.
  tangent to circle, 75.
  tangent to sphere, 395.
Straight lines, divided proportionally, 149.
Subtend, to, 76.
Subtended arc, 76.
Supplement of an angle, 3.
Supplementary angles, 3.
Supplementary dihedral angles, 291.
Surface, area of, 193.
  conical, 377.
  cylindrical, 367.
  defined, 1.
  spherical, 393.
  unit of, 193.
Surfaces, 261.
Symbols, 6.
Symmetrical polyhedral angles, 306.
Symmetrical spherical triangles, 409.
Symmetry, axis of, 65.
  center of, 65.
  defined, 65.

Tangent, circles, 76.
  defined, 75.
  spheres, 395.
Tetrahedron, defined, 313.
  regular, 354.
Theorem, converse of, 20.
  defined, 8.
  elements of, 19.
  of limits, 98.
Third proportional, 143.
Total area, of cone, 377.
  of cylinder, 367.
  of frustum of cone, 378.
  of prism, 314.
  of pyramid, 332.
Transversal, 22.
Trapezium, 47.
Trapezoid, 47.
Triangle, acute, 5.
  altitude of, 37.
  angles of, 4.
  base of, 5.
Triangle, defined, 4.
  equiangular, 5.
  equilateral, 5.
  exterior angle of, 39.
  isosceles, 5.
  median of, 37.
  obtuse, 5.
  right, 5.
  scalene, 5.
  sides of, 4.
  spherical, 408.
  vertex of, 5.
  vertices of, 4.
Triangles, similar, 154.
Triangular prism, 314.
Triangular pyramid, 332.
Trihedral angle, birectangular, 307.
  defined, 307.
  isosceles, 307.
  rectangular, 307.
  trirectangular, 307.
Trirectangular spherical triangle, 408.
Trirectangular trihedral angle, 307.
Truncated prism, 314.
Truncated pyramid, 333.

Unit, of measure, 96.
  of spherical triangle, 408.
  of surface, 193.
  of volume, 315.

Variable, defined, 96.
  limit of, 96, 97.
Vertex, of angle, 2.
  of polyhedral angle, 305.
  of pyramid, 332.
  of spherical pyramid, 428.
  of triangle, 5.
Vertex angle, 5.
Vertical angles, 2.
Vertical dihedral angles, 291.
Vertical polyhedral angles, 306.
Vertices, of lune, 427.
  of polygon, 60.
  of polyhedron, 313.
  of quadrilateral, 47.
  of spherical triangle, 408.
  of triangle, 4.
Volume, 315.

Wedge, spherical, 429.

Zone, 428.

4.99